STOCHASTICALLY STABLE GLOBALLY COUPLED MAPS WITH BISTABLE THERMODYNAMIC LIMIT

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Abstract. We study systems of globally coupled interval maps, where the identical individual maps have two expanding, fractional linear, onto branches, and where the coupling is introduced via a parameter - common to all individual maps - that depends in an analytic way on the mean field of the system. We show: 1) For the range of coupling parameters we consider, finite-size coupled systems always have a unique invariant probability density which is strictly positive and analytic, and all finite-size systems exhibit exponential decay of correlations. 2) For the same range of parameters, the self-consistent Perron-Frobenius operator which captures essential aspects of the corresponding infinite-size system (arising as the limit of the above when the system size tends to infinity), undergoes a supercritical pitchfork bifurcation from a unique stable equilibrium to the coexistence of two stable and one unstable equilibrium.

1. Introduction

Globally coupled maps are collections of individual discrete-time dynamical systems (their *units*) which act independently on their respective phase spaces, except for the influence (the *coupling*) of a common parameter that is updated, at each time step, as a function of the *mean field* of the whole system. Systems of this type have received some attention through the work of Kaneko [9, 10] in the early 1990s, who studied systems of N quadratic maps acting on coordinates $x_1, \dots, x_N \in [0, 1]$, and coupled by a parameter depending in a simple way on $\bar{x} := N^{-1}(x_1 + \cdots + x_N)$. His key observation, for huge *system size* N, was the following: if $(\bar{x}^t)_{t=0,1,2,\dots}$ denotes the time series of mean field values of the system started in a random configuration $(x_1, \dots, x_N)$, then, for many parameters of the quadratic map, and even for very small coupling strength, pairs $(\bar{x}^t, \bar{x}^{t+1})$ of consecutive values of the field showed complicated functional dependencies plus some noise of order $N^{-1/2}$, whereas for uncoupled systems of the same size the $\bar{x}^t$, after a while,

Date: December 21, 2008.

2000 *Mathematics Subject Classification.* 37A60,37D99,37L60,82C20.

Key words and phrases. globally coupled maps, mean field, self consistent Perron Frobenius operator, bifurcation, bistability, iterated function system, Pick-Herglotz-Nevanlinna functions.

This cooperation was supported by the DFG grant Ke-514/7-1 (Germany). J.-B.B. was also partially supported by CNRS (France). The authors acknowledge the hospitality of the ESI (Austria) where part of this research was done. G.K. thanks Carlangelo Liverani for a discussion that helped to shape the ideas in Section 5.3.

are constant up to some noise of order $N^{-1/2}$. While the latter observation is not surprising for independent units, the complicated dependencies for weakly coupled systems, a phenomenon Kaneko termed *violation of the law of large numbers*, called for closer investigation.

The rich bifurcation structure of the family of individual quadratic maps may offer some explanations, but since a mathematically rigorous investigation of even a small number of coupled quadratic maps in the chaotic regime still is a formidable task, there seem to be no serious attempts to tackle this problem.

A model which is mathematically much easier to treat is given by coupled tent maps. Indeed, for tent maps with slope larger than $\sqrt{2}$ and moderate coupling strength, a system of N mean field coupled units has an ergodic invariant probability density with exponentially decreasing correlations [13]. This is true for all N and for coupling strengths that can be chosen to be the same for all N. Nevertheless, Ershov and Potapov [7] showed numerically that (albeit on a much smaller length scale than in the case of coupled quadratic maps) also mean field coupled tent maps exhibit a violation of the law of large numbers in the aforementioned sense. They also provided a mathematical analysis which demonstrated that the discontinuities of the invariant density of a tent map are at the heart of the problem. Their analysis was not completely rigorous, however, as Chawanya and Morita [2] could show that there are indeed (exceptional) parameters of the system for which there is no violation of the law of large numbers - contrary to the predictions in [7]. On the other hand, references [17, 18] contain further simulation results on systems violating the law of large numbers. (But at present, a mathematically rigorous treatment of globally coupled tent maps that is capable of classifying and explaining the diverse dynamical effects that have been observed does not seem to be in sight either.) These studies were complemented by papers by Järvenpää [8] and Keller [12], showing (among other things) that globally coupled systems of smooth expanding circle maps do not display violation of the law of large numbers at small coupling strength, because their invariant densities are smooth.

Given this state of knowledge, the present paper investigates specific systems of globally coupled piecewise fractional linear maps on the interval $X := [-\frac{1}{2}, \frac{1}{2}]$, where each individual map has a smooth invariant density. For small coupling strength, Theorem 4 in [12] extends easily to this setup and proves the absence of a violation of the law of large numbers. For larger coupling strength, however, we are going to show that this phenomenon does occur in the following sense:

Bifurcation: The nonlinear *self-consistent Perron-Frobenius operator (PFO)* $\widetilde{P}$ on $L_1(X, \lambda)$, which describes the dynamics of the system in its thermodynamic limit, undergoes a supercritical pitchfork bifurcation as the coupling strength increases. (Here and in the sequel λ denotes Lebesgue measure.)

Mixing: At the same time, all corresponding finite-size systems have unique absolutely continuous invariant probability measures $\boldsymbol{\mu}_N$ on their N-dimensional state space, and exhibit exponential decay of correlations under this measure.

Stable behaviour: In the *stable regime*, i.e. for fixed small coupling strength below the bifurcation point of the infinite-size system, the measures $\boldsymbol{\mu}_N$ converge weakly, as the system size $N \to \infty$, to an infinite product measure $(u_0 \cdot \lambda)^{\mathbb{N}}$, where u_0 is the unique fixed point of $\widetilde{P}$.

Bistable behaviour: In the *bistable regime*, i.e. for fixed coupling strength above the bifurcation point of the infinite-size system, all possible weak limits of the measures $\boldsymbol{\mu}_N$ are convex combinations of the three infinite product measures $(u_r \cdot \lambda)^{\mathbb{N}}$, $r \in \{-r_*, 0, r_*\}$, where now u_0 is the unique unstable fixed point of $\widetilde{P}$ and $u_{\pm r_*}$ are its two stable fixed points. (We conjecture that the measure $(u_0 \cdot \lambda)^{\mathbb{N}}$ is not charged in the limit.)

This scenario clearly bears some resemblance to the Curie-Weiss model from statistical mechanics and its dynamical variants.

We also stress that a simple modification of our system leads to a variant where, instead of two stable fixed points, one stable two-cycle for $\widetilde{P}$ is created at the bifurcation point. This may be viewed as the simplest possible scenario for a violation of the law of large numbers in Kaneko's original sense.

In the next section we describe our model in detail, and formulate the main results. Section 3 contains the proofs for finite-size systems. In Section 4 we start the investigation of the infinite-size system via the self-consistent PFO $\widetilde{P}$. We observe that this operator preserves a class of probability densities which can be characterised as derivatives of Herglotz-Pick-Nevanlinna functions. Integral representations of these functions reveal a hidden order structure, which is respected by the operator $\widetilde{P}$, and allows us to describe the pitchfork bifurcation. In Section 5 this dynamical picture for $\widetilde{P}$ is extended to arbitrary densities. Finally, in Section 6, we discuss the situation when some noise is added to the dynamics.

2. Model and main results

2.1. **The parametrised family of maps.** Throughout, all measures are understood to be Borel, and we let $\mathsf{P}(B) := \{\text{probability measures on } B\}$. Lebesgue measure will be denoted by λ. We introduce a 1-parameter family of piecewise fractional-linear transformations T_r on $X := [-\frac{1}{2}, \frac{1}{2}]$, which will play the role of the local maps. To facilitate manipulation of such maps, we use their standard matrix representation, letting

$$f_M(x) := \frac{ax+b}{cx+d} \quad \text{for any real } 2 \times 2\text{-matrix } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

so that $f'_M(x) = (ad - bc)/(cx + d)^2$ and $f_M \circ f_N = f_{MN}$. Specifically, we consider the function f_{M_r}, depending on a parameter $r \in (-2, 2)$, given by the coefficient matrix

$$M_r := \begin{pmatrix} r+4 & r+1 \\ 2r & 2 \end{pmatrix}.$$

One readily checks that $f_{M_r}(-\frac{1}{2}) = -\frac{1}{2}$, $f_{M_r}(\frac{1}{2}) = \frac{3}{2}$, $f_{M_r}(\alpha_r) = \frac{1}{2}$ for $\alpha_r := -r/4$, and that (the infimum being attained on ∂X)

$$f'_{M_r}(x) = \frac{4 - r^2}{2\,(rx+1)^2} \geqslant 2\,\frac{2 - |r|}{2 + |r|} = \inf_X f'_{M_r} > 0 \quad \text{ for } x \in X.$$

The latter shows that f_{M_r} is uniformly expanding if and only if $|r| < \frac{2}{3}$, and we define our *single-site maps* $T_r : X \to X$ with parameter $r \in (-2/3, 2/3)$ by letting

$$T_r(x) := f_{M_r}(x) \bmod \left(\mathbb{Z} + \frac{1}{2}\right) = \begin{cases} f_{M_r}(x) & \text{on } [-\frac{1}{2}, \alpha_r), \\ f_{M_r}(x) - 1 = f_{N_r}(x) & \text{on } (\alpha_r, \frac{1}{2}], \end{cases}$$

where

$$N_r := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} M_r \; .$$

We thus obtain a family $(T_r)_{r\in(-2/3,2/3)}$ of uniformly expanding, piecewise invertible maps $T_r : X \to X$, each having two increasing covering branches. Note also that this family is symmetric in that

$$-\,T_r(-x) = T_{-r}(x) \quad \text{ for } r \in \left(-\tfrac{2}{3}, \tfrac{2}{3}\right) \text{ and } x \in X. \tag{2.1}$$

According to well-known folklore results, each map T_r, $r \in (-2/3, 2/3)$, has a unique invariant probability density $u_r \in \mathcal{D} := \{u \in L_1(X, \lambda) : u \geqslant 0, \int_X u\, d\lambda = 1\}$, and T_r is exact (hence ergodic) w.r.t. the corresponding invariant measure. Due to (2.1), we have $u_{-r}(x) = u_r(-x) \bmod \lambda$. We denote the *Perron-Frobenius operator* (*PFO*), w.r.t. Lebesgue measure λ, of a map T by P_T, abbreviating $P_r := P_{T_r}$. In our construction below we will exploit the fact that 2-to-1 fractional linear maps like T_r in fact enable a fairly explicit analysis of their PFOs on a suitable class of densities. In particular, the u_r are known explicitly:

Remark 1. Let $\gamma_r := \frac{r}{1+r}$, $\delta_r := \frac{r}{1-r}$. Then

$$\tilde{u}_r(x) := \int_{\gamma_r}^{\delta_r} \frac{1}{(1-xy)^2}\, dy = \frac{2r^2}{(rx - (1-r))(rx - (1+r))} \tag{2.2}$$

is an integrable invariant density for T_r, see [21]. Its normalised version

$$u_r(x) := \left(\log \frac{r^2 - 4}{9r^2 - 4}\right)^{-1} \cdot \tilde{u}_r(x) \tag{2.3}$$

is the unique T_r-invariant probability density.

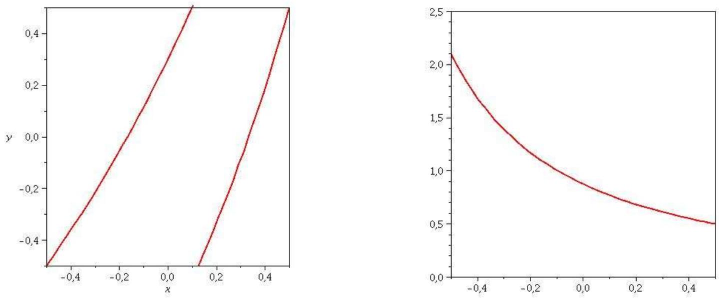

FIGURE 1. The functions T_r (left), and u_r (right), for $r = -\frac{1}{2}$.

The key point in the choice of this family of maps is that for $r < 0$, T_r is steeper in the positive part of X than in its negative part, hence typical orbits spend more time on the negative part, which is confirmed by the invariant density (see Figure 1). If $r > 0$, then T_r favours the positive part.

The heuristics of our construction is that for sufficiently strong coupling this effect of "polarisation" is reinforced and gives rise to bistable behaviour.

2.2. **The field and the coupling.** For any probability measure $Q \in \mathsf{P}(X)$, we denote its mean by

$$\phi(Q) := \int_X x \, dQ(x), \tag{2.4}$$

and call this the *field* of Q. With a slight abuse of notation we also write, for $u \in \mathcal{D}$,

$$\phi(u) := \int_X x u(x) \, dx = \phi(Q) \quad \text{if } u = \frac{dQ}{d\lambda}, \tag{2.5}$$

and, for $\mathbf{x} \in X^N$,

$$\phi(\mathbf{x}) := \frac{1}{N} \sum_{i=1}^{N} x_i = \phi(Q) \quad \text{if } Q = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \,. \tag{2.6}$$

To define the system of globally coupled maps (both in the finite- and the infinite-size case) we will, at each step of the iteration, determine the actual parameter as a function of the present field. This is done by means of a *feedback function* $G : X \to R := [-0.4, 0.4]$ which we always assume to be real-analytic [1] and *S-shaped* in that it satisfies $G'(x) > 0$ and $G(-x) = -G(x)$ for all $x \in X$, while $G''(x) < 0$ if $x > 0$. The most important single parameter in our model is going to be $B := G'(0)$ which quantifies the *coupling strength*.

[1]This is only required to obtain highest regularity of the invariant densities of the finite-size systems in Theorem 1. Everything else remains true if G is merely of class $\mathcal{C}^2$.

Remark 2. The following will be our standard example of a suitable feedback function G:

$$G(x) := A \tanh\left(\frac{B}{A}x\right), \tag{2.7}$$

where $0 < A \leqslant 0.4$ and $0 \leqslant B \leqslant 18$. (This requires some numerical effort. For $0 < A \leqslant 0.2$ and $0 \leqslant B \leqslant 15$, elementary estimates suffice.)

For the results to follow we shall impose a few additional constraints on the feedback function G, made precise in Assumptions I and II below.

2.3. **The finite-size systems.** We consider a system $\mathbf{T}_N : X^N \to X^N$ of N coupled copies of the parametrised map, defined by $(\mathbf{T}_N(\mathbf{x}))_i = T_{r(\mathbf{x})}(x_i)$ with $r(\mathbf{x}) := G(\phi(\mathbf{x}))$. For the following theorem, which we prove in section 3, we need the following assumption (satisfied by the example above):

(2.8) **Assumption I:** $G'(x) \leqslant 25 - 50|G(x)|$ for all $x \in X$.

Theorem 1 (**Ergodicity and mixing of finite-size systems**). *Suppose the S-shaped function G satisfies* (2.8). *Then, for any $N \in \mathbb{N}$, the map $\mathbf{T}_N : X^N \to X^N$ has a unique absolutely continuous invariant probability measure $\boldsymbol{\mu}_N$. Its density is strictly positive and real analytic. The systems $(\mathbf{T}_N, \boldsymbol{\mu}_N)$ are exponentially mixing in various strong senses, in particular do Hölder observables have exponentially decreasing correlations.*

The key to the proof is an estimate ensuring uniform expansion. After establishing the latter in Section 3, the theorem follows from "folklore" results whose origins are not so easy to locate in the literature. In a C^2-setting, existence, uniqueness and exactness of an invariant density were proved essentially by Krzyzewski and Szlenk [15]. Exponential mixing follows from the compactness of the transfer operator first observed by Ruelle [20]. For a result which applies in our situation and entails Theorem 1, we refer to the main theorem of [16].

2.4. **The self-consistent PFO and the thermodynamic limit of the finite-size systems.** Since the coupling we defined is of mean-field type, we can adapt from the probabilistic literature (see for example [22, 4]) the classical method of taking the *thermodynamic limit* of our family of finite-size systems $\mathbf{T}_N$, as $N \to \infty$. To do so, consider the set $\mathsf{P}(X)$ of Borel probability measures on X, equipped with the topology of weak convergence and the resulting Borel σ-algebra on $\mathsf{P}(X)$. Define $\widetilde{T} : \mathsf{P}(X) \to \mathsf{P}(X)$ by

$$\widetilde{T}(Q) := Q \circ T_{r(Q)}^{-1}, \quad \text{where} \quad r(Q) := G(\phi(Q)). \tag{2.9}$$

We can then represent the evolution of any finite-size system using $\widetilde{T}$. Indeed, if $\epsilon_N(\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ is the empirical measure of $\mathbf{x} = (x_i)_{1 \leqslant i \leqslant N}$, then $\epsilon_N : X^N \to \mathsf{P}(X)$ satisfies $\epsilon_N \circ \mathbf{T}_N = \widetilde{T} \circ \epsilon_N$.

Furthermore, when restricted to the set of probability measures absolutely continuous with respect to λ, $\tilde{T}$ is represented by the *self-consistent Perron Frobenius operator*, which is the nonlinear positive operator $\widetilde{P}$ defined as

$$\widetilde{P} : L_1(X,\lambda) \to L_1(X,\lambda), \quad \widetilde{P}u := P_{G(\phi(u))}u. \tag{2.10}$$

Clearly, this map satisfies $\widetilde{T}(u \cdot \lambda) = (\tilde{P}u) \cdot \lambda$ and preserves the set $\mathcal{D}$ of probability densities. Note, however, that it does not contract, i.e. there are $u, v \in \mathcal{D}$ such that $\|\widetilde{P}u - \widetilde{P}v\|_{L_1(X,\lambda)} > \|u - v\|_{L_1(X,\lambda)}$.

One may finally join these two aspects, the action of $\widetilde{T}$ on means of Dirac masses, or on absolutely continuous measures, via the following observation:

Proposition 1 (**Propagation of chaos**). *Let $Q = u \cdot \lambda \in \mathsf{P}(X)$, with $u \in \mathcal{D}$. If $(x_i)_{i\geqslant 1}$ is chosen according to $Q^{\otimes \mathbb{N}}$, then, for any $n \geqslant 0$, the empirical measures $\epsilon_N(\mathbf{T}_N^n(x_1,\dots,x_N))$ converge weakly to $(\tilde{P}^n u) \cdot \lambda$ as $N \to \infty$.*

This result confirms the point of view that the self-consistent PFO $\tilde{P}$ represents the infinite-size thermodynamic limit $N \to \infty$ of the finite-size systems $\mathbf{T}_N$. Its proof is reasonably simple (easier than for stochastic evolutions). The only difficulty is that $\tilde{T}$ is not a continuous map on the whole of $\mathsf{P}(X)$. This can be overcome with the following lemma, which Proposition 1 is a direct consequence of, and whose proof is given in Section A.1.

Lemma 1 (**Continuity of $\tilde{T}$ at non-atomic measures**). *Assume that a sequence $(Q_n)_{n\geqslant 1}$ in $\mathsf{P}(X)$ converges weakly to some non-atomic Q. Then $(\widetilde{T}Q_n)_{n\geqslant 1}$ converges weakly to $\widetilde{T}Q$.*

Here is an immediate consequence of this lemma that will be used below.

Corollary 1. *Assume that a sequence $(\pi_n)_{n\geqslant 1}$ of $\widetilde{T}$-invariant Borel probability measures* on $\mathsf{P}(X)$ *converges weakly to some probability* π on $\mathsf{P}(X)$. *If there is a Borel set $A \subseteq \mathsf{P}(X)$ with $\pi(A) = 1$ which only contains non-atomic measures, then π is also $\widetilde{T}$-invariant.*

2.5. **The long-term behaviour of the infinite-size system.** Our goal is to analyse the asymptotics of $\widetilde{P}$ on $\mathcal{D}$. Some basic features of $\widetilde{P}$ can be understood considering the dynamics of

$$H : \left(-\tfrac{2}{3}, \tfrac{2}{3}\right) \to R := \left[-\tfrac{4}{10}, \tfrac{4}{10}\right], \quad H(r) := G(\phi(u_r)),$$

which governs the action of $\widetilde{P}$ on the densities u_r introduced in § 2.1, as

$$\widetilde{P}u_r = P_{H(r)}u_r. \tag{2.11}$$

In studying $\widetilde{P}$, we will always presuppose the following:

$$\textbf{Assumption II:} \quad H \text{ is S-shaped.} \tag{2.12}$$

This assumption can be checked numerically for specific feedback functions G, like that of Remark 2, cf. §A.2 below. By (2.1), $H(-r) = -H(r)$. Note, however, that $r \mapsto \phi(u_r)$ itself is not S-shaped (see Figure 3) so that the

S-shapedness of G alone is not sufficient for that of H.

Assumption II will enter our arguments only via the following dichotomy which it entails: either

$$\begin{gathered} H(r) \text{ has a unique fixed point at } r=0 \\ \text{(the \textbf{stable regime} with } H'(0) \leqslant 1 \text{ and } r=0 \text{ stable),} \\ \text{or} \\ H(r) \text{ has exactly three fixed points } -r_* < 0 < r_* \\ \text{(the \textbf{bistable regime} with } H'(0) > 1 \text{ and } \pm r_* \text{ stable).} \end{gathered}$$

We will see that $H' > 0$ and $H'(0) = G'(0)/6$, so that the stable regime corresponds to the condition $G'(0) \leqslant 6$. Observe now that

$$\widetilde{P} u_r = u_r \quad \text{iff} \quad \begin{cases} r = 0 & \text{(in the stable regime)} \\ r \in \{0, \pm r_*\} & \text{(in the bistable regime)} \end{cases} \tag{2.13}$$

(since $u_r \neq u_{r'}$ for $r \neq r'$, and each T_r is ergodic). We are going to show that the fixed points $u_0 = 1_X$, and $u_{\pm r_*}$ dominate the long-term behaviour of $\widetilde{P}$ on $\mathcal{D}$ completely, and that they inherit the stability properties of the corresponding parameters $-r_* < 0 < r_*$. Therefore, the stable/bistable terminology for H introduced above also provides an appropriate description of the asymptotic behaviour of $\widetilde{P}$.

Theorem 2 (**Long-term behaviour of $\widetilde{P}$ on $\mathcal{D}$**). *Consider $\widetilde{P} : \mathcal{D} \to \mathcal{D}$, $\mathcal{D}$ equipped with the metric inherited from $L_1(X, \lambda)$. Assuming (I) and (II), we have the following:*

1) In the stable regime, u_0 is the unique fixed point of $\widetilde{P}$, and attracts all densities, that is,

$$\lim_{n\to\infty} \widetilde{P}^n u = u_0 \quad \text{for all } u \in \mathcal{D}.$$

2) In the bistable regime, $\{u_{-r_*}, u_0, u_{r_*}\}$ are the only fixed points of $\widetilde{P}$. Now u_0 is unstable, while u_{-r_*} and u_{r_*} are stable. More precisely:
   - *a) $u_{\pm r_*}$ are stable fixed points for $\widetilde{P}$ in the sense that their respective basins of attraction are L_1-open.*
   - *b) If $u \in \mathcal{D}$ is not attracted by u_{-r_*} or u_{r_*}, then it is attracted by u_0.*
   - *c) u_0 is not stable. Indeed, u_0 can be L_1-approximated by convex analytic densities from either basin. It is a hyperbolic fixed point of $\widetilde{P}$ in the sense made precise in Proposition 5 of Section 5.*

Example 1. In case $G(x) = A \tanh(Bx/A)$ with $0 < A \leqslant 0.4$ and $0 \leqslant B \leqslant 18$, both theorems apply. The infinite-size system is stable iff $B \leqslant 6$, and bistable otherwise, while all finite-size systems have a unique a.c.i.m. in this parameter region.

The theorem summarises the contents of Propositions 3, 4 and 5 of Section 5 (which, in fact, provide more detailed information). The proofs rest on the fact that PFOs of maps with full fractional-linear branches leave the class of Herglotz-Pick-Nevanlinna functions invariant. This observation can be used to study the action of $\widetilde{P}$ in terms of an iterated function system on the interval $[-2,2]$ with two fractional-linear branches and place dependent probabilities. In the bistable regime the system is of course not contractive, but it has strong monotonicity properties and special geometric features which allow to prove the theorem.

Our third theorem, which is essentially a corollary to the previous ones, describes the passage from finite-size systems to the infinite-size system. Below, weak convergence of the $\boldsymbol{\mu}_N \in \mathsf{P}(X^N)$ to some $\boldsymbol{\mu} \in \mathsf{P}(X^{\mathbb{N}})$ means that $\int \varphi \, d\boldsymbol{\mu}_N \to \int \varphi \, d\boldsymbol{\mu}$ for all continuous $\varphi : X^{\mathbb{N}} \to \mathbb{R}$ which only depend on finitely many coordinates. (So that $\int \varphi \, d\boldsymbol{\mu}_N$ is defined, in the obvious fashion, for N large enough.)

Theorem 3 (From finite to infinite size – the limit as $N \to \infty$). *The $\mathbf{T}_N$-invariant probability measures $\boldsymbol{\mu}_N$ of Theorem 1 correspond to the $\widetilde{T}$-invariant probability measures $\boldsymbol{\mu}_N \circ \epsilon_N^{-1}$ on $\mathsf{P}(X)$. All weak accumulation points π of the latter sequence are $\widetilde{T}$-invariant probability measures concentrated on the set of measures absolutely continuous w.r.t. λ. Furthermore:*

1) *In the stable regime, the sequence $(\boldsymbol{\mu}_N \circ \epsilon_N^{-1})_{N \geqslant 1}$ converges weakly to the point mass δ_λ. In other words, the sequence $(\boldsymbol{\mu}_N)_{N\geqslant 1}$ converges weakly to the pure product measure $\lambda^{\mathbb{N}}$ on $X^{\mathbb{N}}$.*
2) *In the bistable regime, each weak accumulation point π of the sequence $(\boldsymbol{\mu}_N \circ \epsilon_N^{-1})_{N \geqslant 1}$ is of the form $\alpha\, \delta_{u_{-r_*}\lambda} + (1-2\alpha)\, \delta_{u_0\lambda} + \alpha\, \delta_{u_{r_*}\lambda}$ for some $\alpha \in [0, \frac{1}{2}]$. In other words, each weak accumulation point of the sequence $(\boldsymbol{\mu}_N)_{N\geqslant 1}$ is of the form $\alpha(u_{-r_*}\lambda)^{\mathbb{N}} + (1-2\alpha)\lambda^{\mathbb{N}} + \alpha(u_{r_*}\lambda)^{\mathbb{N}}$.*

Remark 3. We cannot prove, so far, that $\alpha = \frac{1}{2}$, which is to be expected because u_0 is an unstable fixed point of $\widetilde{P}$. In Section 6 we show that $\alpha = \frac{1}{2}$ indeed, if some small noise is added to the system.

Proof of Theorem 3. As $\epsilon_N \circ \mathbf{T}_N = \widetilde{T} \circ \epsilon_N$, the $\mathbf{T}_N$-invariant probability measures $\boldsymbol{\mu}_N$ of Theorem 1 correspond to $\widetilde{T}$-invariant probability measures $\boldsymbol{\mu}_N \circ \epsilon_N^{-1}$ on $\mathcal{P}(X)$. Their possible weak accumulation points are all concentrated on sets of measures from $\mathcal{P}(X)$ with density w.r.t. λ, see Theorem 3 in [12]. (The proof of that part of the theorem we refer to does not rely on the continuity of the local maps that is assumed in that paper.) Therefore Corollary 1 shows that all these accumulation points are $\widetilde{T}$-invariant probability measures concentrated on measures with density w.r.t. λ. In other words, they can be interpreted as $\widetilde{P}$-invariant probability measures on $\mathcal{D}$. Now Theorem 2 implies that the sequence $(\boldsymbol{\mu}_N \circ \epsilon_N^{-1})_{N \geqslant 1}$ converges weakly to the point mass $\delta_{u_0\lambda}$ in the stable regime, whereas, in the bistable regime, each such limit measure is of the form $\alpha\, \delta_{u_{-r_*}\lambda} + (1-2\alpha)\, \delta_{u_0\lambda} + \alpha\, \delta_{u_{r_*}\lambda}$ for some $\alpha \in [0, \frac{1}{2}]$ (observe the symmetry of the system). Now the corresponding assertions on the measures $\boldsymbol{\mu}_N$ follow along known lines, for a reference see e.g. [12, Proposition 1]. $\square$

3. Proofs: the finite-size systems

We assume throughout this section that

$$|G(x)| \leqslant 0.5 \text{ and } G'(x) \leqslant 25 - 50|G(x)| \text{ for all } |x| \leqslant \frac{1}{2}. \tag{3.1}$$

In order to apply the main theorem of Mayer [16] we must check his assumptions (A1) – (A4) for the map $\mathbf{T} = \mathbf{T}_N$. To that end define $\mathbf{F} : X^N \to [-\frac{1}{2}, \frac{3}{2}]^N$ by $(\mathbf{F}(\mathbf{x}))_i = f_{M_{r(\mathbf{x})}}(x_i)$. Obviously $\mathbf{T}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \bmod \left(\mathbb{Z} + \frac{1}{2}\right)^N$, and (A1) – (A4) follow readily from the following facts that we are going to prove:

Lemma 2. $\mathbf{F} : X^N \to [-\frac{1}{2}, \frac{3}{2}]^N$ *is a homeomorphism which extends to a diffeomorphism between open neighbourhoods of* X^N *and* $[-\frac{1}{2}, \frac{3}{2}]^N$.

Lemma 3. *The inverse* $\mathbf{F}^{-1}$ *of* $\mathbf{F}$ *is real analytic and can be continued to a holomorphic mapping on a complex* δ*-neighbourhood* Ω *of* $[-\frac{1}{2}, \frac{3}{2}]^N$ *such that* $\mathbf{T}^{-1}(\Omega)$ *is contained in a* δ'*-neighbourhood of* X^N *for some* $0 < \delta' < \delta$.

To verify these two lemmas we need the following uniform expansion estimate which we will prove at the end of this section. (Here $\|.\|$ denotes the Euclidean norm.)

Lemma 4 (**Uniform expansion**). *There is a constant* $\rho \in (0, 1)$ *such that* $\|(D\mathbf{F}(\mathbf{x}))^{-1}\| \leqslant \rho$ *for all* $N \in \mathbb{N}$ *and* $\mathbf{x} \in X^N$.

Proof of Lemma 2. Obviously $\mathbf{F}(X^N) \subseteq [-\frac{1}{2}, \frac{3}{2}]^N$. Hence it is sufficient to prove the assertions of the lemma for the map $\widetilde{\mathbf{F}} := \frac{1}{2}(\mathbf{F} - (\frac{1}{2}, \ldots, \frac{1}{2})^{\intercal}) : X^N \to X^N$. As each f_{M_r} is differentiable on $(-1, 1)$ (recall that $|r| < \frac{2}{3}$), $\widetilde{\mathbf{F}}$ extends to an analytic mapping from $(-1, 1)^N \to \mathbb{R}^N$. By Lemma 4, it is locally invertible on $\Omega_\varepsilon := (-\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)^N$ for each sufficiently small $\varepsilon \geqslant 0$. (Note that $\Omega_0 = \text{int}(\mathrm{X})$.) All we need to show is that this implies global invertibility of $\widetilde{\mathbf{F}}|_{\Omega_0} : \Omega_0 \to \Omega_0$, because then the possibility to extend $\widetilde{\mathbf{F}}$ diffeomorphically to a small open neighbourhood of X^N in $\mathbb{R}^N$ follows again from the local invertibility on Ω_ε for some $\varepsilon > 0$.

So we prove the global invertibility of $\widetilde{\mathbf{F}}|_{\Omega_0} : \Omega_0 \to \Omega_0$. As each $\tilde{f}_{M_r} := \frac{1}{2}(f_{M_r} - \frac{1}{2}) : X \to X$ is a homeomorphism that leaves fixed the endpoints of the interval X, we have $\widetilde{\mathbf{F}}(\partial X^N) \subseteq \partial X^N$ and $\widetilde{\mathbf{F}}(\Omega_0) \subseteq \Omega_0$. Observing the simple fact that Ω_0 is a paracompact connected smooth manifold without boundary and with trivial fundamental group, we only need to show that $\widetilde{\mathbf{F}}|_{\Omega_0} : \Omega_0 \to \Omega_0$ is proper in order to deduce from [3, Corollary 1] that $\widetilde{\mathbf{F}}|_{\Omega_0}$ is a diffeomorphism of Ω_0. So let K be a compact subset of Ω_0. As $\widetilde{\mathbf{F}}|_{\Omega_0}$ extends to the continuous map $\widetilde{\mathbf{F}} : X^N \to X^N$ and as $\widetilde{\mathbf{F}}(\partial X^N) \subseteq \partial X^N$, the set $\widetilde{\mathbf{F}}|_{\Omega_0}^{-1}(K) = \widetilde{\mathbf{F}}^{-1}(K) \subset \mathrm{int}(\mathrm{X}^{\mathrm{N}}) \subseteq X^N$ is closed and hence compact. Therefore $\widetilde{\mathbf{F}}|_{\Omega_0} : \Omega_0 \to \Omega_0$ is indeed proper. □

Proof of Lemma 3. As $\mathbf{F}$ is real analytic on a real neighbourhood of X^N, the real analyticity of $\mathbf{F}^{-1}$ on a real neighbourhood of $[-\frac{1}{2}, \frac{3}{2}]^N$ follows from the real analytic inverse function theorem [14, Theorem 18.1]. It extends to a holomorphic function on a complex δ-neighbourhood Ω of $[-\frac{1}{2}, \frac{3}{2}]^N$ – see e.g. the discussion of complexifications of real analytic maps in [14, pp.162-163]. If $\delta > 0$ is sufficiently small, Lemma 4 implies that $\mathbf{F}^{-1}$ is a uniform contraction on Ω. Hence the δ' in the statement of the lemma can be chosen strictly smaller than δ. □

Proof of Lemma 4. Recall that $(\mathbf{F}(\mathbf{x}))_i = f_{r(\mathbf{x})}(x_i)$ where $r(\mathbf{x}) = G(\phi(\mathbf{x}))$, $\phi(\mathbf{x}) = \frac{1}{N}\sum_{i=1}^N x_i$ and we write f_r instead of f_{M_r}. Denote $g(\mathbf{x}) = G'(\phi(\mathbf{x}))$,

$$\begin{aligned}
\Delta_1(\mathbf{x}) &:= \mathrm{diag}(f_r'(x_1), \dots, f_r'(x_N)) \\
\Delta_2(\mathbf{x}) &:= \mathrm{diag}(\frac{\partial f_r}{\partial r}(x_1), \dots, \frac{\partial f_r}{\partial r}(x_N)) \\
E_N &:= \begin{pmatrix} \frac{1}{N} \cdots \frac{1}{N} \\ \vdots \quad\ \vdots \\ \frac{1}{N} \cdots \frac{1}{N} \end{pmatrix}
\end{aligned}$$

and observe that $q(x) := (4 - r^2)\frac{\partial f_r}{\partial r}(x)/f_r'(x)$ simplifies to $q(x) = 1 - 4x^2$ so that

$$\Delta_1(\mathbf{x})^{-1}\Delta_2(\mathbf{x}) = \frac{1}{4 - r^2}\,\mathrm{diag}\,(q(x_1), \dots, q(x_N)) =: \frac{1}{4 - r^2}\Delta_3(\mathbf{x}) \ .$$

Then the derivative of the coupled map $\mathbf{F}(\mathbf{x}) = (f_{r(\mathbf{x})}(x_1), \dots, f_{r(\mathbf{x})}(x_N))$ is

$$D\mathbf{F}(\mathbf{x}) = \Delta_1(\mathbf{x}) + \Delta_2(\mathbf{x})E_N g(\mathbf{x}) = \Delta_1(\mathbf{x})\left(\mathbf{1} + \frac{1}{4 - r^2}\Delta_3(\mathbf{x})E_N g(\mathbf{x})\right) \ ,$$

with $\mathbf{1}$ denoting the identity matrix. Letting

$$\begin{aligned}
\mathbf{q} = \mathbf{q}(\mathbf{x}) &:= (q(x_1), \dots, q(x_N))^{\mathsf{T}} \\
\mathbf{e}_N &:= (\frac{1}{N}, \dots, \frac{1}{N})^{\mathsf{T}}
\end{aligned}$$

and observing that $\Delta_3(\mathbf{x}) = \mathrm{diag}(\mathbf{q}(\mathbf{x}))$ so that $\Delta_3(\mathbf{x})E_N = \mathbf{q}\,\mathbf{e}_N^{\mathsf{T}}$, the inverse of $D\mathbf{F}(\mathbf{x})$ is

$$D\mathbf{F}(\mathbf{x})^{-1} = \left(\mathbf{1} - \frac{g(\mathbf{x})}{4 - r^2 + g(\mathbf{x})\mathbf{e}_N^{\mathsf{T}}\mathbf{q}(\mathbf{x})}\mathbf{q}(\mathbf{x})\mathbf{e}_N^{\mathsf{T}}\right)\Delta_1(\mathbf{x})^{-1} \ .$$

In order to check conditions under which $\mathbf{F}$ is uniformly expanding in all directions, it is sufficient to find conditions under which $\|D\mathbf{F}(\mathbf{x})^{-1}\| < 1$ *uniformly* in $\mathbf{x}$. Observe first that $\|\Delta_1(\mathbf{x})^{-1}\| \leqslant (\inf f_r')^{-1} \leqslant \frac{1}{2}\frac{2+|r|}{2-|r|} < 1$ for $|r| < \frac{2}{3}$. From now on we fix a point $\mathbf{x}$ and suppress it as an argument to all functions. Then, if $\mathbf{v}$ is any vector in $\mathbb{R}^N$, some scalar multiple of it can be decomposed in a unique way as $\alpha\mathbf{v} = \mathbf{q} - \mathbf{p}$ where $\mathbf{p}$ is perpendicular to $\mathbf{q}$.

Denote $\bar{p} = \mathbf{e}_N^\mathsf{T}\mathbf{p}$, $\bar{q} = \mathbf{e}_N^\mathsf{T}\mathbf{q}$, and observe that $\bar{q} \geqslant 0$ as $\mathbf{q}$ has only nonnegative entries. We estimate the euclidian norm of $(\mathbf{1} - \frac{g}{4-r^2+g\mathbf{e}_N^\mathsf{T}\mathbf{q}}\mathbf{q}\,\mathbf{e}_N^\mathsf{T})(\alpha\mathbf{v})$:

$$\begin{aligned}
&\left\| \left(\mathbf{1} - \frac{g}{4-r^2+g\mathbf{e}_N^\mathsf{T}\mathbf{q}}\mathbf{q}\,\mathbf{e}_N^\mathsf{T} \right)(\alpha\mathbf{v}) \right\|^2 \\
&= \left(1 + \frac{g\bar{p} - g\bar{q}}{4-r^2+g\bar{q}} \right)^2 \|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 \\
&= \left(1 + \frac{\bar{p} - \bar{q}}{\Gamma + \bar{q}} \right)^2 \|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 \\
&= \left(\frac{1 + \Gamma^{-1}\bar{p}}{1 + \Gamma^{-1}\bar{q}} \right)^2 \|\mathbf{q}\|^2 + \|\mathbf{p}\|^2
\end{aligned} \tag{3.2}$$

where $\Gamma := \frac{4-r^2}{g}$. As $\bar{p} \leqslant N^{-1/2}\|\mathbf{p}\|$ and $\bar{q} \geqslant N^{-1}\|\mathbf{q}\|^2$ (observe that all entries of $\mathbf{q}$ are bounded by 1), we can continue the above estimate with

$$\leqslant \|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 + 2\|\mathbf{p}\|\|\mathbf{q}\|\frac{\Gamma^{-1}N^{-1/2}\|\mathbf{q}\|}{(1+\Gamma^{-1}N^{-1}\|\mathbf{q}\|^2)^2} + \|\mathbf{p}\|^2\frac{\Gamma^{-2}N^{-1}\|\mathbf{q}\|^2}{(1+\Gamma^{-1}N^{-1}\|\mathbf{q}\|^2)^2}$$

To estimate this expression we abbreviate temporarily $t := N^{-1/2}\|\mathbf{q}\|$. Then $0 \leqslant t \leqslant 1$, and straightforward maximisation yields:

$$\begin{aligned}
\frac{\Gamma^{-1}N^{-1/2}\|\mathbf{q}\|}{(1+\Gamma^{-1}N^{-1}\|\mathbf{q}\|^2)^2} &\leqslant \frac{9}{16\sqrt{3}\sqrt{\Gamma}} \\
\frac{\Gamma^{-2}N^{-1}\|\mathbf{q}\|^2}{(1+\Gamma^{-1}N^{-1}\|\mathbf{q}\|^2)^2} &\leqslant \frac{1}{4\Gamma}
\end{aligned}$$

So we can continue the above estimate by

$$(3.2) \leqslant (\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2)\left(1 + \frac{9}{16\sqrt{3}\sqrt{\Gamma}} + \frac{1}{4\Gamma}\right)$$

where $\|\mathbf{q}\|^2 + \|\mathbf{p}\|^2 = \|\alpha\mathbf{v}\|^2$. Hence

$$\left\| \left(\mathbf{1} - \frac{g}{1-r+g\mathbf{e}_N^\mathsf{T}\mathbf{q}}\mathbf{q}\mathbf{e}_N^\mathsf{T} \right)(\alpha\mathbf{v}) \right\| \leqslant \left(1 + \frac{9}{16\sqrt{3}\sqrt{\Gamma}} + \frac{1}{4\Gamma}\right)^{1/2} \|\alpha\mathbf{v}\|$$

and therefore

$$\|D\mathbf{F}^{-1}(x)\| \leqslant \rho := \left(\frac{1}{2}\cdot\frac{2+|r|}{2-|r|}\right)\left(1 + \frac{9}{16\sqrt{3}\sqrt{\Gamma}} + \frac{1}{4\Gamma}\right)^{1/2}.$$

Observing $\Gamma = \frac{4-r^2}{g}$ one finds numerically that the norm is bounded by 0.99396 uniformly for all x, if $-0.5 \leqslant r \leqslant 0.5$ and $0 \leqslant g \leqslant 25 - 50r$. Hence the map $\mathbf{F}$ is uniformly expanding in all directions provided (3.1) holds. $\square$

4. An iterated function system representation for smooth densities

4.1. **An invariant class of densities.** The PFOs P_r allow a detailed analysis since their action on certain densities has a convenient explicit description: Consider the family $(w_y)_{y\in(-2,2)}$ of probability densities on X given by

$$w_y(x) := \frac{1-y^2/4}{(1-xy)^2}, \quad x \in X.$$

As pointed out in [21] (using different parametrisations), Perron-Frobenius operators P_{f_M} of fractional-linear maps f_M act on these densities via their *duals* $f_{M^\#}$, where

$$M^\# := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \quad \text{for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

in that

$$P_{f_M}(1_J \cdot w_y) = \left(\int_J w_y \, d\lambda\right) \cdot w_{y'} \quad \text{with } y' = f_{M^\#}(y), \tag{4.1}$$

for matrices M and intervals $J \subseteq X$ for which $f_M(J) = X$. (This can also be verified by direct calculation). Since $f_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)}(x) = \frac{1}{x}$, one can compute the duals

$$\begin{aligned} \sigma_r(y) &:= f_{M_r^\#}(y) = \frac{1}{f_{M_r}(\frac{1}{y})} = \frac{2(y+r)}{(r+1)y+r+4} \quad \text{and} \\ \tau_r(y) &:= f_{N_r^\#}(y) = \frac{1}{f_{N_r}(\frac{1}{y})} = \frac{\sigma_r(y)}{1-\sigma_r(y)} = \frac{2(y+r)}{(r-1)y-r+4} \end{aligned} \tag{4.2}$$

of the individual branches of T_r, then express $P_r\, w_y$ as the convex combination

$$\begin{aligned} P_r\, w_y &= P_{f_{M_r}}(1_{[-\frac{1}{2},\alpha_r)} \cdot w_y) + P_{f_{N_r}}(1_{(\alpha_r,\frac{1}{2}]} \cdot w_y) \\ &= p_r(y) \cdot w_{\sigma_r(y)} + (1-p_r(y)) \cdot w_{\tau_r(y)}, \end{aligned} \tag{4.3}$$

with weights

$$p_r(y) := \int_{-1/2}^{\alpha_r} w_y(x)\, dx = \frac{1}{2} - \frac{r+y}{4+ry} \quad \text{and} \quad 1-p_r(y) = \frac{1}{2} + \frac{r+y}{4+ry} \tag{4.4}$$

It is straightforward to check that for every $r \in (-2,2)$ the functions σ_r, τ_r are continuous and strictly increasing on $[-2,2]$ with images $\sigma_r([-2,2]) = [-2,2/3]$ and $\tau_r([-2,2]) = [-2/3,2]$.

From this remark and (4.3) it is clear that the P_r preserve the class of those $u \in \mathcal{D}$ which are convex combinations $u = \int_{(-2,2)} w_y\, d\mu(y) = \int w_\bullet\, d\mu$ of the special densities w_y for some *representing measure* μ from $\mathsf{P}(-2,2)$.

We find that P_r acts on representing measures according to

$$P_r\left(\int_{(-2,2)} w_\bullet\, d\mu\right) = \int_{(-2,2)} w_\bullet\, d\left(\mathcal{L}_r^*\mu\right) \quad \text{with} \tag{4.5}$$
$$\mathcal{L}_r^*\mu := (p_r \cdot \mu)\circ\sigma_r^{-1} + ((1-p_r)\cdot\mu)\circ\tau_r^{-1},$$

where $p_r \cdot \mu$ denotes the measure with density p_r w.r.t. μ.

To continue, we need to collect several facts about the dual maps σ_r and τ_r. We have

$$\sigma_r'(y) = \frac{2(4-r^2)}{(ry+y+r+4)^2} \quad \text{and} \quad \tau_r'(y) = \frac{2(4-r^2)}{(ry-y-r+4)^2}, \tag{4.6}$$

showing that σ_r and τ_r are strictly concave, respectively convex, on $[-2,2]$. One next gets readily from (4.2) that $1/\sigma_r - 1/\tau_r = 1$ wherever defined, and

$$\sigma_r(y) < \tau_r(y) \text{ for } y \in [-2,2]\diagdown\{-r\}$$

while (and this observation will be crucial later on) σ_r and τ_r have a common zero $z_r := -r$ and

$$\sigma_r'(z_r) = \tau_r'(z_r) = \frac{2}{4-r^2}, \quad \text{and} \quad \sigma_r(z_r+t) = \frac{2\,t}{(r+1)\,t+4-r^2}\,. \tag{4.7}$$

In the following, we restrict our parameter r to the set $R = \left[-\frac{4}{10}, \frac{4}{10}\right]$. Direct calculation proves that, letting $Y := \left[-\frac{2}{3}, \frac{2}{3}\right]$, we have

$$\sigma_r(Y) \cup \tau_r(Y) \subseteq Y \quad \text{if } r \in R, \tag{4.8}$$

so that Y is an invariant set for all such σ_r and τ_r, and that

$$\sup_Y \sigma_r' = \sigma_r'\left(-\frac{2}{3}\right) \leqslant \frac{3}{4} \quad \text{and} \quad \sup_Y \tau_r' = \tau_r'\left(\frac{2}{3}\right) \leqslant \frac{3}{4} \quad \text{for } r \in R, \tag{4.9}$$

which provides us with a common contraction rate on Y for the σ_r and τ_r from this parameter region. All these features of σ_r and τ_r are illustrated by Figure 2.

We denote by $\overline{w}(y) := \phi(w_y)$ the field of the density w_y, and find by explicit integration that

$$\overline{w}(y) = \left(\frac{1}{4} - \frac{1}{y^2}\right)\log\frac{1+y/2}{1-y/2} + \frac{1}{y} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)}\left(\frac{y}{2}\right)^{2k+1} \tag{4.10}$$

for $y \in Y$. In particular, $\overline{w}(0) = 0$ and $\overline{w}'(y) \geqslant \frac{1}{6} > 0$, so the field depends monotonically on y. Note also that we have, for all $\mu \in \mathsf{P}(Y)$,

$$\phi\left(\int_Y w_\bullet\, d\mu\right) = \int_Y \int_X x w_y(x)\, dx\, d\mu(y) = \int_Y \overline{w}\, d\mu. \tag{4.11}$$

We focus on densities u with representing measure μ supported on Y, i.e. on the class $\mathcal{D}' := \{u = \int_Y w_\bullet\, d\mu : \mu \in \mathsf{P}(Y)\}$. Writing

$$r_\mu := G\left(\int_Y \overline{w}\, d\mu\right) = G(\phi(u)) \in R, \tag{4.12}$$

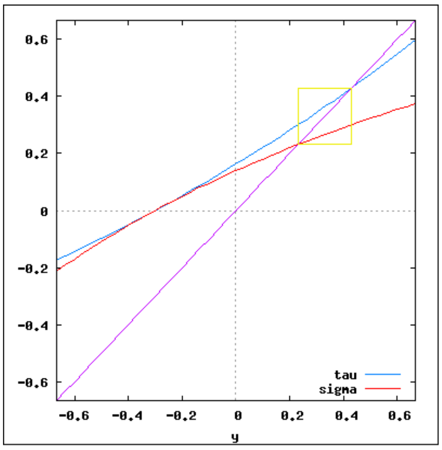

FIGURE 2. The dual maps σ_r and τ_r for $r = 0.3$. The small invariant box has endpoints γ_r and τ_r.

and recalling (4.5), we find that our nonlinear operator $\widetilde{P}$ acts on the representing measures from $\mathsf{P}(Y)$ via

$$\widetilde{P}\left(\int_Y w_\bullet \, d\mu\right) = \int_Y w_\bullet \, d\left(\widetilde{\mathcal{L}}^*\mu\right) \qquad \text{with} \tag{4.13}$$

$$\widetilde{\mathcal{L}}^*\mu := \mathcal{L}^*_{r_\mu}\mu = (p_{r_\mu}\cdot\mu)\circ\sigma_{r_\mu}^{-1} + ((1-p_{r_\mu})\cdot\mu)\circ\tau_{r_\mu}^{-1}.$$

As $\operatorname{supp}(\widetilde{\mathcal{L}}^*\mu)$, the support of $\widetilde{\mathcal{L}}^*\mu$, is contained in $\sigma_r(\operatorname{supp}(\mu))\cup\tau_r(\operatorname{supp}(\mu))$, it is immediate from (4.8) that

$$\widetilde{P}\mathcal{D}' \subseteq \mathcal{D}'.$$

For $r \in R = [-\frac{4}{10}, \frac{4}{10}]$ we find that σ_r and τ_r each have a unique stable fixed point in Y, given by

$$\sigma_r(\gamma_r) = \gamma_r := \frac{r}{r+1} \qquad \text{and} \qquad \tau_r(\delta_r) = \delta_r := \frac{-r}{r-1},$$

respectively. Note that the interval $Y_r := [\gamma_r, \delta_r]$ of width $2r^2/(1-r^2)$ between these stable fixed points is invariant under both σ_r and τ_r, see the small boxed region in Figure 2. Furthermore, each of γ_r and δ_r is mapped to r under the branch not fixing it, i.e.

$$\sigma_r(\delta_r) = \tau_r(\gamma_r) = r,$$

meaning that, restricted to Y_r, σ_r and τ_r are the inverse branches of some 2-to-1 piecewise fractional linear map $S_r : Y_r \to Y_r$.

The explicit T_r-invariant densities u_r from (2.3) can be represented as $u_r = \int_Y w_\bullet \, d\mu_r$ with $\mu_r \in \mathsf{P}(Y_r) \subseteq \mathsf{P}(Y)$ given by

$$\frac{d\mu_r}{d\lambda}(y) = \left(\log\frac{r^2-4}{9r^2-4}\right)^{-1} \frac{1_{Y_r}(y)}{1-y^2/4}. \tag{4.14}$$

Our goal in this section is to study the asymptotic behaviour of $\widetilde{P}$ on $\mathcal{D}'$, using its representation by means of the IFS $\tilde{\mathcal{L}}^*$. We will prove

Proposition 2 (**Long-term behaviour of** $\widetilde{P}$ **on** $\mathcal{D}'$). *Take any $u \in \mathcal{D}'$, $u = \int_Y w_y \, d\mu(y)$ for some $\mu \in \mathsf{P}(Y)$. The following is an exhaustive list of possibilities for the asymptotic behaviour of the sequence $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$:*

(1) $\tilde{\mathcal{L}}^{*n}\mu \succ \delta_0$ for some $n \geqslant 0$. Then $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges to μ_{r_*} and hence $\tilde{P}^n u$ converges to u_{r_*} in $L_1(X,\lambda)$.

(2) The interval $\mathrm{conv}(\mathrm{supp}(\tilde{\mathcal{L}}^{*n}\mu))$ contains 0 for all $n \geqslant 0$. Then $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges to δ_0 and hence $\widetilde{P}^n u$ converges to u_0 in $L_1(X,\lambda)$. In this case also the length of $\mathrm{conv}(\mathrm{supp}(\tilde{\mathcal{L}}^{*n}\mu))$ tends to 0

(3) $\tilde{\mathcal{L}}^{*n}\mu \prec \delta_0$ for some $n \geqslant 0$. Then $(\tilde{\mathcal{L}}^{*n}\mu)_n$ converges to μ_{-r_*} and hence $\tilde{P}^n u$ converges to u_{-r_*} in $L_1(X,\lambda)$.

In the stable regime, only scenario (2) is possible, so that we always have convergence of $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ to δ_0.

Our arguments will rely on continuity and monotonicity properties of the IFS, that we detail below, before proving Proposition 2 in Section 4.4.

4.2. **The IFS: continuity.** *Convergence in* $\mathsf{P}(Y)$, $\lim \mu_n = \mu$, will always mean weak convergence of measures, $\int_Y \rho \, d\mu_n \to \int_Y \rho \, d\mu$ for bounded continuous $\rho : Y \to \mathbb{R}$. Since Y is a bounded interval, this is equivalent to convergence in the *Wasserstein-metric* d_W on $\mathsf{P}(Y)$. If F_μ and F_ν are the distribution functions of μ and ν, then

$$\mathsf{d}_W(\mu,\nu) := \int_{-\infty}^{\infty} |F_\mu(x) - F_\nu(x)| \, dx. \tag{4.15}$$

The Kantorovich-Rubinstein theorem (e.g. [6, Ch.11]) provides an additional characterisation:

$$\mathsf{d}_W(\mu,\nu) = \sup_{\psi:\, \mathrm{Lip}_Y[\psi]\leqslant 1} \int_Y \psi \, d(\mu-\nu) \tag{4.16}$$

for any $\mu,\nu \in \mathsf{P}(Y)$. Here, $\mathrm{Lip}_Y[\psi] := \sup_{y,y'\in Y; y\neq y'} |\psi(y) - \psi(y')| \,/\, |y-y'|$ for any $\psi : Y \to \mathbb{R}$ (and analogously for functions on other domains). We now see that there is a constant $K > 0$ (the common Lipschitz bound for the functions $y \mapsto w_y(x)$ on Y, where $x \in X$) such that

$$\left\| \int_Y w_\bullet \, d\mu - \int_Y w_\bullet \, d\nu \right\|_{L_1(X,\lambda)} \leqslant K \cdot \mathsf{d}_W(\mu,\nu). \tag{4.17}$$

This means that, for densities from $\mathcal{D}'$, convergence of the representing measures implies L_1-convergence of the densities.

We will also use the following estimate.

Lemma 5 (**Continuity of** $(r,\mu) \mapsto \mathcal{L}_r^*\mu$). *There are constants $\kappa_1, \kappa_2 > 0$ such that*

$$\mathsf{d}_W(\mathcal{L}_r^*\mu, \mathcal{L}_s^*\nu) \leqslant \kappa_1 \mathsf{d}_W(\mu,\nu) + \kappa_2 |r-s|$$

for all $\mu, \nu \in \mathsf{P}(Y)$ and all $r, s \in R$.

Proof. For Lipschitz functions ψ on Y we have

$$\begin{aligned}\int_Y \psi \, d\,(\mathcal{L}_r^*\mu - \mathcal{L}_s^*\nu) &\leqslant \mathrm{Lip}_Y\,[(\psi\circ\sigma_r)\,p_r + (\psi\circ\tau_r)(1-p_r)]\cdot \mathsf{d}_W(\mu,\nu) \\ &\quad + \sup_{y\in Y}\mathrm{Lip}_R\,[\psi(\sigma_\cdot(y))\,p_\cdot(y) + \psi(\tau_\cdot(y))(1-p_\cdot(y))]\cdot|r-s| \end{aligned} \tag{4.18}$$

Suppose that ψ is $\mathcal{C}^1$, with $|\psi'| \leqslant 1$. Then the first Lipschitz constant is bounded by

$$\kappa_1 := \sup_{r\in R}\left[\left\|(\tau_r-\sigma_r)\cdot p_r'\right\|_\infty + \left\|\sigma_r'\cdot p_r + \tau_r'\cdot(1-p_r)\right\|_\infty\right] < \infty,$$

and the second one by

$$\kappa_2 := \sup_{r\in R}\left[\left\|\frac{\partial\sigma_r}{\partial r}\cdot p_r\right\|_\infty + \left\|\frac{\partial\tau_r}{\partial r}\cdot(1-p_r)\right\|_\infty + \left\|(\tau_r-\sigma_r)\cdot\frac{\partial p_r}{\partial r}\right\|_\infty\right] < \infty,$$

and the lemma follows from the Kantorovich-Rubinstein theorem (4.16), since these $\mathcal{C}^1$ functions ψ uniformly approximate the Lipschitz functions appearing there. □

Corollary 2. *The operators $\mathcal{L}_r^*$, $r \in R$, and $\tilde{\mathcal{L}}^*$ are uniformly Lipschitz-continuous on $\mathsf{P}(Y)$ for the Wasserstein metric.*

Proof. The Lipschitz-continuity of $\mathcal{L}_r^*$ is immediate from Lemma 5. For $\tilde{\mathcal{L}}^*$, we recall from (2.9) and (4.13) that $\tilde{\mathcal{L}}^*\mu = \mathcal{L}_{r_\mu}^*\mu$ with $r_\mu = G(\int_Y \overline{w}\,d\mu)$ so that

$$|r_\mu - r_\nu| \leqslant \mathrm{Lip}(G)\ \mathrm{Lip}(\overline{w})\ \mathsf{d}_W(\mu,\nu),$$

which allows to conclude with Lemma 5. □

Remark 4. Rigorous numerical bounds give $\kappa_1 \leqslant 0.5761$ and $\kappa_2 \leqslant 0.5334$. These estimates can be used to show that not only the individual $\mathcal{L}_r^*$ are uniformly contracting on $\mathsf{P}(Y)$, but also (under suitable restrictions on the function G) $\tilde{\mathcal{L}}^*$ is a uniform contraction on $\mathsf{P}(Y^*)$ where Y^* is a suitable neighbourhood of the support of μ_{r_*}, and μ_{r_*} is the representing measure of u_{r_*} with r_* the unique positive fixed point of the equation $r = G(\phi(u_r))$. Our treatment of $\tilde{\mathcal{L}}^*$, however, does not rely on these estimates, because it is based on monotonicity properties explained below.

4.3. **The IFS: monotonicity.** On the space $\mathsf{P}(Y)$ of probability measures μ,ν representing densities from $\mathcal{D}'$, we introduce an order relation by defining

$$\mu \preceq \nu \quad :\Leftrightarrow \quad \forall y\in Y:\ \mu(y,\infty) \leqslant \nu(y,\infty) \tag{4.19}$$

The symbols $\prec$, $\succeq$ and $\succ$ designate the usual variants of $\preceq$. We collect a few elementary facts on this order relation:

(4.20) $\mu \preceq \nu$ if and only if $\int_Y u\,d\mu \leqslant \int_Y u\,d\nu$ for each bounded and non-decreasing $u : Y \to \mathbb{R}$.

In particular, if $\mu \preceq \nu$, then $\int_Y \overline{w}\, d\mu \leqslant \int_Y \overline{w}\, d\nu$, and hence $r_\mu \leqslant r_\nu$ as well.

(4.21) If $\mu \preceq \nu$ and if $\rho_1, \rho_2 : Y \to Y$ are non-decreasing and such that $\rho_1(y) \leqslant \rho_2(y)$ for all y, then $\mu \circ \rho_1^{-1} \preceq \nu \circ \rho_2^{-1}$.

(4.22) If $\mu \preceq \nu$ and if $\int_Y u\, d\mu = \int_Y u\, d\nu$ for some strictly increasing $u : Y \to \mathbb{R}$, then $\mu = \nu$.

(4.23) Let $z \in Y$. Then $\delta_z \preceq \mu$ if and only if $\mathrm{supp}(\mu) \subseteq [z, \infty)$.

We also observe that the representing measures μ_r of the T_r-invariant densities u_r form a linearly ordered subset of $\mathsf{P}(Y)$. Routine calculations based on (4.14) show that

$$\mu_r \prec \mu_s \quad \text{if } r < s. \tag{4.24}$$

Our analysis of the asymptotic behaviour of the IFS will crucially depend on the fact that the operators $\mathcal{L}_r^*$ and $\tilde{\mathcal{L}}^*$ respect this order relation on $\mathsf{P}(Y)$, as made precise in the next lemma:

Lemma 6 (**Monotonicity of** $(r, \mu) \mapsto \mathcal{L}_r^* \mu$)**.** *Let $\mu, \nu \in \mathsf{P}(Y)$ and $r, s \in R$.*
a) If $\mu \prec \nu$, then $\mathcal{L}_r^* \mu \prec \mathcal{L}_r^* \nu$.
b) If $r < s$, then $\mathcal{L}_r^* \mu \prec \mathcal{L}_s^* \mu$.
c) If $\mu \prec \nu$, then $\tilde{\mathcal{L}}^* \mu \prec \tilde{\mathcal{L}}^* \nu$.

Proof. Let $u : Y \to \mathbb{R}$ be bounded and non-decreasing and recall that

$$\int_Y u\, d(\mathcal{L}_r^* \mu) = \int_Y [u \circ \sigma_r \cdot p_r + u \circ \tau_r \cdot (1 - p_r)]\, d\mu. \tag{4.25}$$

a) Let $\mu \preceq \nu$. In view of (4.20) we can prove $\mathcal{L}^* \mu \preceq \mathcal{L}^* \nu$ by showing that the integrand on the right-hand side of (4.25) is non-decreasing. For this we use the facts that σ_r and τ_r are strictly increasing with $\sigma_r < \tau_r$, and that p_r is non-increasing, since $p_r'(y) = -\frac{4-r^2}{(4+ry)^2} < 0$. One gets then, for $x<y$,

$$\begin{aligned}
&u(\sigma_r x) p_r(x) + u(\tau_r x)(1 - p_r(x)) \\
=&u(\sigma_r x) p_r(y) + u(\sigma_r x) \underbrace{(p_r(x) - p_r(y))}_{>0} + u(\tau_r x)(1 - p_r(x)) \\
\leqslant&u(\sigma_r x) p_r(y) + u(\tau_r x)(p_r(x) - p_r(y)) + u(\tau_r x)(1 - p_r(x)) \\
=&u(\sigma_r x) p_r(y) + u(\tau_r x)(1 - p_r(y)) \\
=&u(\sigma_r y) p_r(y) + u(\tau_r y)(1 - p_r(y)) \\
&\quad - \underbrace{\left[(u(\sigma_r y) - u(\sigma_r x))\, p_r(y) + (u(\tau_r y) - u(\tau_r x))\, (1 - p_r(y)) \right]}_{\geqslant 0}.
\end{aligned} \tag{4.26}$$

Hence $\mu \preceq \nu$ implies $\mathcal{L}_r^* \mu \preceq \mathcal{L}_r^* \nu$. Now, if u is strictly increasing, then (4.26) always is a strict inequality, i.e. the integrand on the right-hand side of (4.25) is strictly increasing. Therefore, $\mu \prec \nu$ implies $\mathcal{L}_r^* \mu \prec \mathcal{L}_r^* \nu$ by (4.22).

b) We must show that (4.25) is non-decreasing as a function of r. To this end note first that

$$\frac{\partial p_r(y)}{\partial r} = \frac{y^2-4}{(ry+4)^2} < 0 \ , \tag{4.27}$$

$$\frac{\partial \sigma_r(y)}{\partial r} = \frac{8-2y^2}{(ry+y+r+4)^2} > 0 \ , \tag{4.28}$$

$$\frac{\partial \tau_r(y)}{\partial r} = \frac{8-2y^2}{(ry-y-r+4)^2} > 0 \ . \tag{4.29}$$

Hence, if $r < s$, then

$$\begin{aligned} & u(\sigma_r x)p_r(x) + u(\tau_r x)(1-p_r(x)) \\ =& u(\sigma_r x)p_s(x) + u(\sigma_r x)\underbrace{(p_r(x)-p_s(x))}_{>0} + u(\tau_r x)(1-p_r(x)) \\ \leqslant& u(\sigma_s x)p_s(x) + u(\tau_s x)(p_r(x)-p_s(x)) + u(\tau_s x)(1-p_r(x)) \\ =& u(\sigma_s x)p_s(x) + u(\tau_s x)(1-p_s(x)) \ , \end{aligned}$$

and for strictly increasing u we have indeed a strict inequality.

c) This follows from a) and b):

$$\tilde{\mathcal{L}}^*\mu = \mathcal{L}^*_{r_\mu}\mu \prec \mathcal{L}^*_{r_\mu}\nu \preceq \mathcal{L}^*_{r_\nu}\nu = \tilde{\mathcal{L}}^*\nu$$

as $r_\mu = G(\int_Y \overline{w}\, d\mu) \leqslant G(\int_Y \overline{w}\, d\nu) = r_\nu$ by (4.20). □

In §5.2 below, we will also make use of a more precise quantitative version of statement a). It is natural to state and prove it at this point.

Lemma 7 (**Quantifying the growth of $\mu \mapsto \mathcal{L}^*_r\mu$**). *Suppose that $\alpha, \beta > 0$ are such that $u' \geqslant \alpha$ and $\tau_r', \sigma_r' \geqslant \beta$. Then, for $\mu \preceq \nu$,*

$$\int_Y u\, d(\mathcal{L}^*_r\nu) - \int_Y u\, d(\mathcal{L}^*_r\mu) \geqslant \alpha\,\beta \left(\int_Y \mathrm{id}\; d\nu - \int_Y \mathrm{id}\; d\mu\right) . \tag{4.30}$$

Proof. Observing that $u(\sigma_r y) - u(\sigma_r x)$ and $u(\tau_r y) - u(\tau_r x)$ are $\geqslant \alpha\beta(y-x)$, we find for the last expression in (4.26) that

$$\left[(u(\sigma_r y) - u(\sigma_r x))\, p_r(y) + (u(\tau_r y) - u(\tau_r x))\,(1-p_r(y))\right] \geqslant \alpha\beta y - \alpha\beta x.$$

This turns (4.26) into a chain of inequalities which shows that the function given by $v(x) := u(\sigma_r x)p_r(x) + u(\tau_r x)(1-p_r(x)) - \alpha\beta x$ is non-decreasing. Hence, by (4.20), $\mu \preceq \nu$ entails $\int_Y v\, d\mu \leqslant \int_Y v\, d\nu$, which is (4.30). □

4.4. **Dynamics of the IFS and the asymptotics of $\widetilde{P}$ on $\mathcal{D}'$.** We are now going to clarify the asymptotic behaviour of $\tilde{\mathcal{L}}^*$ on $\mathsf{P}(Y)$. In view of (4.17), this also determines the asymptotics of $\widetilde{P}$ on $\mathcal{D}'$, and hence proves Proposition 2.

Our argument depends on monotonicity properties which we can exploit since the topology of weak convergence on $\mathsf{P}(Y)$, conveniently given by the

Wasserstein metric, is consistent with the order relation introduced above. Indeed, one easily checks:

(4.31) If (ν_n) and $(\bar{\nu}_n)$ are weakly convergent sequences in $\mathsf{P}(Y)$ with $\nu_n \preceq \bar{\nu}_n$ for all n, then $\lim \nu_n \preceq \lim \bar{\nu}_n$.

Recall from § 2.5 that, in the bistable regime, r_* is the unique positive fixed point of the equation $r = G(\phi(u_r))$. For convenience, we now let

$$r_* := 0 \quad \text{in the stable regime.}$$

Then, in either case, u_r with representing measure μ_r is fixed by $\widetilde{P}$ iff $r \in \{0, \pm r_*\}$. By (4.24) we have $\mu_{-r_*} \preceq \mu_0 = \delta_0 \preceq \mu_{r_*}$ with strict inequalities in the bistable regime.

Lemma 8 (**Convergence by monotonicity**). *If $\mu \preceq \tilde{\mathcal{L}}^*\mu$, then $\mu \preceq \tilde{\mathcal{L}}^*\mu \preceq \tilde{\mathcal{L}}^{*2}\mu \preceq \dots$, and the sequence $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges weakly to a measure $\mu_r \succeq \mu$ with $r \in \{0, \pm r_*\}$. The same holds for $\succeq$ instead of $\preceq$.*

Proof. The monotonicity of the sequence $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ follows immediately from Lemma 6c). Because of (4.31), it implies that the sequence can have at most one weak accumulation point. Compactness of $\mathsf{P}(Y)$ and continuity of $\tilde{\mathcal{L}}^*$ therefore ensure that $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges to a fixed point of $\tilde{\mathcal{L}}^*$, i.e. to one of the measures μ_r with $r \in \{0, \pm r_*\}$, and (4.31) entails $\mu_r \succeq \mu$. The proof for decreasing sequences is the same. □

The following lemma strengthens the previous one considerably. It provides uniform control, in terms of the Wasserstein distance (4.15), on the asymptotics of large families of representing measures.

Lemma 9 (**Convergence by comparison**). *We have the following:*

a) In the stable regime, there exists a sequence $(\varepsilon_n)_{n\geqslant 0}$ of positive real numbers converging to zero such that

$$\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu, \delta_0) \leqslant \varepsilon_n \quad \text{for } \mu \in \mathsf{P}(Y) \text{ and } n \in \mathbb{N}.$$

b) In the bistable regime, for every $y > 0$ there exists a sequence $(\varepsilon_n)_{n\geqslant 0}$ of positive real numbers converging to zero such that

$$\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu, \mu_{r_*}) \leqslant \varepsilon_n \quad \text{for } \mu \in \mathsf{P}(Y) \text{ with } \mu \succeq \delta_y \text{ and } n \in \mathbb{N}.$$

An analogous assertion holds for measures $\mu \preceq \delta_{-y}$.

Proof. As $Y = [-\frac{2}{3}, \frac{2}{3}]$, we trivially have $\delta_{-2/3} \preceq \mu \preceq \delta_{2/3}$ for all $\mu \in \mathsf{P}(Y)$. In particular, $\tilde{\mathcal{L}}^*\delta_{2/3} \preceq \delta_{2/3}$, and Lemma 8 ensures that $(\tilde{\mathcal{L}}^{*n}\delta_{2/3})_{n\geqslant 0}$ converges. Due to Lemma 6c), we have $\delta_0 \preceq \mu_{r_*} = \tilde{\mathcal{L}}^{*n}\mu_{r_*} \preceq \tilde{\mathcal{L}}^{*n}\delta_{2/3}$ for all $n \geqslant 0$, showing, via (4.31), that $\lim \tilde{\mathcal{L}}^{*n}\delta_{2/3} = \mu_{r_*}$. In the same way one proves that $(\tilde{\mathcal{L}}^{*n}\delta_{-2/3})_{n\geqslant 0}$ converges to μ_{-r_*}. For the stable regime this means that both sequences converge to $\delta_0 = \mu_0$.

a) Assume we are in the stable regime. By the above discussion,

$$\varepsilon_n := \mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\delta_{-2/3}, \delta_0) + \mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\delta_{2/3}, \delta_0)$$

tends to zero. For any μ, (4.31) guarantees $\tilde{\mathcal{L}}^{*n}\delta_{-2/3} \preceq \tilde{\mathcal{L}}^{*n}\mu \preceq \tilde{\mathcal{L}}^{*n}\delta_{2/3}$ for all $n \geqslant 0$. Hence $F_{\tilde{\mathcal{L}}^{*n}\delta_{-2/3}}(y) \geqslant F_{\tilde{\mathcal{L}}^{*n}\mu}(y) \geqslant F_{\tilde{\mathcal{L}}^{*n}\delta_{2/3}}(y)$ for all y, proving $\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu, \delta_0) \leqslant \varepsilon_n$.

b) Now consider the bistable regime. Note first that if there is a suitable sequence $(\varepsilon_n)_{n\geqslant 0}$ for some $y > 0$, then it also works for all $y' > y$. Therefore, there is no loss of generality if we assume that $y > 0$ is so small that

$$\sigma_{G(\overline{w}(y))}(y) = \left(\frac{1}{2} + \frac{G'(0)}{12}\right) y + \mathcal{O}(y^2) > y \tag{4.32}$$

(use (4.10), (4.28) and (4.2) to see that this can be achieved.) Since $\mathcal{L}_r^*\delta_y = p_r(y)\delta_{\sigma_r(y)} + (1 - p_r(y))\delta_{\tau_r(y)}$ and $\sigma_r(y) < \tau_r(y)$, we then have $\delta_0 \prec \delta_y \prec \mathcal{L}^*_{G(\overline{w}(y))}\delta_y = \tilde{\mathcal{L}}^*\delta_y$, recall Lemma 6. Lemma 8 then implies that $(\tilde{\mathcal{L}}^{*n}\delta_y)_{n\geqslant 0}$ converges to μ_{r_*}. In view of the initial discussion, $(\tilde{\mathcal{L}}^{*n}\delta_{2/3})_{n\geqslant 0}$ converges to μ_{r_*} as well, so that

$$\varepsilon_n := \mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\delta_y, \mu_{r_*}) + \mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\delta_{2/3}, \mu_{r_*})$$

defines a sequence of reals converging to zero. Now take any $\mu \in \mathsf{P}(Y)$ with $\mu \succeq \delta_y$, then $\tilde{\mathcal{L}}^{*n}\delta_{2/3} \succeq \tilde{\mathcal{L}}^{*n}\mu \succeq \tilde{\mathcal{L}}^{*n}\delta_y$ for all $n \geqslant 0$, and $\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu, \mu_{r_*}) \leqslant \varepsilon_n$ follows as in the proof of a) above. □

This observation enables us to determine the asymptotics of $\tilde{\mathcal{L}}^{*n}\mu$ for any $\mu \in \mathsf{P}(Y)$ which is completely supported on the positive half $(0, 2/3]$ of Y (meaning that $\mu \succ \delta_0$, cf. (4.23)), or on its negative half $[-2/3, 0)$.

Corollary 3. *Let $\mu \in \mathsf{P}(Y)$.*

a) In the stable regime, the sequence $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges to δ_0.

b) In the bistable regime, if $\mu \succ \delta_0$, then the sequence $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ converges to μ_{r_*}. If $\mu \prec \delta_0$, it converges to μ_{-r_*}.

Proof. a) follows immediately from Lemma 9a). We turn to b): Let $r := \int_Y \overline{w}\, d\mu$. Then $r > 0$ because $\overline{w} > 0$ on $(0, 2/3]$ and $\mu \succ \delta_0$. Therefore $\sigma_r(0) > \sigma_0(0) = 0$. Fix some y as in Lemma 9b), w.l.o.g. $y \in (0, \sigma_r(0))$. Then $\delta_0 \prec \delta_y \preceq \mathcal{L}_r^*\mu = \tilde{\mathcal{L}}^*\mu$ since σ_r and τ_r map supp(μ) into $[\sigma_r(0), 2/3]$, so that indeed $\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu, \mu_{r_*}) \to 0$ as $n \to \infty$ by the lemma. □

It remains to investigate the convergence of sequences $(\tilde{\mathcal{L}}^{*n}\mu)_{n\geqslant 0}$ when none of these measures can be compared (in the sense of $\prec$) to δ_0. To this end let $[a_0, b_0] := Y$. Given a sequence of parameters $r_1, r_2, \ldots \in R$ define

$$a_n := \sigma_{r_n} \circ \ldots \circ \sigma_{r_1}(a_0) \quad \text{and} \quad b_n := \tau_{r_n} \circ \ldots \circ \tau_{r_1}(b_0)$$

for $n \geqslant 1$, and, for any $\mu = \mu_0 \in \mathsf{P}(Y) = \mathsf{P}[a_0, b_0]$, consider the measures

$$\mu_n := \mathcal{L}^*_{r_n} \circ \ldots \circ \mathcal{L}^*_{r_1}\mu.$$

Then $\mathrm{supp}(\mu_n) \subseteq \mathrm{supp}(\mathcal{L}^*_{r_n}\mu_{n-1}) \subseteq \sigma_{r_n}([a_{n-1}, b_{n-1}]) \cup \tau_{r_n}([a_{n-1}, b_{n-1}]) \subseteq [a_n, b_n]$ by induction. Write $[a,b]_\varepsilon := [a-\varepsilon, b+\varepsilon]$, where $\varepsilon \geqslant 0$. The next lemma exploits the crucial observation that the two branches σ_r and τ_r have tangential contact at their common zero z_r, see Figure 2.

Lemma 10 (**Support intervals close to zeroes).** *There exists some $C \in (0, \infty)$ such that the following holds: Suppose that $(r_n)_{n\geqslant 1}$ is any given sequence in R. If for some $\varepsilon \geqslant 0$ and $\bar{n}(\varepsilon) \geqslant 0$ we have*

$$(\clubsuit_\varepsilon) \qquad z_{r_{n+1}} \in [a_n, b_n]_\varepsilon \quad \textit{for } n \geqslant \bar{n}(\varepsilon),$$

then

$$\overline{\lim_{n\to\infty}} |b_n - a_n| \leqslant C\varepsilon^2 \ , \tag{4.33}$$

$$\overline{\lim_{n\to\infty}} \max(|a_n|, |b_n|) \leqslant \frac{3}{4}\varepsilon + C\varepsilon^2, \tag{4.34}$$

and, in case $\varepsilon = 0$,

$$0 \in [a_n, b_n] \quad \textit{for } n \geqslant \bar{n}(0) + 1. \tag{4.35}$$

Proof. Let $\varepsilon \geqslant 0$ and assume $(\clubsuit_\varepsilon)$. Note that, for $n \geqslant \bar{n} = \bar{n}(\varepsilon)$,

$$\begin{aligned} &\text{if } a_n > z_{r_{n+1}}, \text{ then } 0 < a_{n+1} < 3/4 \cdot \varepsilon, \\ &\text{if } b_n < z_{r_{n+1}}, \text{ then } -3/4 \cdot \varepsilon < b_{n+1} < 0, \\ &\text{if } a_n \leqslant z_{r_{n+1}} \leqslant b_n, \text{ then } a_{n+1} \leqslant 0 \leqslant b_{n+1}. \end{aligned}$$

The first implication holds because $0 = \sigma_{r_{n+1}}(z_{r_{n+1}}) < \sigma_{r_{n+1}}(a_n) = a_{n+1}$ as $\sigma_{r_{n+1}}$ increases strictly, and since by $(\clubsuit_\varepsilon)$ we have $a_n \in (z_{r_{n+1}}, z_{r_{n+1}} + \varepsilon]$, whence $a_{n+1} < \varepsilon \cdot \sup \sigma'_{r_{n+1}} \leqslant 3\varepsilon/4$ due to (4.9). Analogously for the second implication. The third is immediate from monotonicity.

Now, as σ_r and τ_r share a common zero z_r, (4.9) ensures $b_{\bar{n}+m} - a_{\bar{n}+m} \leqslant \frac{3}{4}(b_{\bar{n}+m-1} - a_{\bar{n}+m-1})$ in case $z_r \in [a_{\bar{n}+m-1}, b_{\bar{n}+m-1}]$. Otherwise, note that z_r is ε-close to one of the endpoints, w.l.o.g. to $a_{\bar{n}+m-1}$. Since σ_r and τ_r are tangent at z_r, there is some $C > 0$ s.t. $0 \leqslant \tau_{r_{\bar{n}+m}}(a_{\bar{n}+m-1}) - \sigma_{r_{\bar{n}+m}}(a_{\bar{n}+m-1}) \leqslant \frac{C}{4}\varepsilon^2$ in this case, while (4.9) controls the rest of $b_{\bar{n}+m} - a_{\bar{n}+m}$. In view of $\mathrm{diam}(\mathrm{Y}) = 4/3$, we thus obtain, for $m \geqslant 1$,

$$b_{\bar{n}+m} - a_{\bar{n}+m} \leqslant \frac{3}{4}(b_{\bar{n}+m-1} - a_{\bar{n}+m-1}) + \frac{C}{4}\varepsilon^2 \leqslant \ldots \leqslant \frac{4}{3}\left(\frac{3}{4}\right)^m + C\varepsilon^2.$$

Statement (4.33) follows immediately. For the asymptotic estimate (4.34) on $\max(|a_n|, |b_n|) = \max(-a_n, b_n)$, use the above inequality plus the observation that, by the first two implications stated in this proof, $a_{\bar{n}+m}$ and $-b_{\bar{n}+m}$ never exceed $3\varepsilon/4$. Finally, if $\varepsilon = 0$, (4.35) is straightforward from $(\clubsuit_\varepsilon)$ and the third implication above. $\square$

While the full strength of this lemma will only be required in the next subsection, the $\varepsilon = 0$ case enables us to now conclude the

Proof of Proposition 2. The conclusions of (1) and (3) follow from Corollary 3. If neither of these two cases applies, then the assumption of (2) must be satisfied, and so condition ($\clubsuit_0$) of Lemma 10 is satisfied with $\bar{n}(0) = 0$. Hence $\lim_{n\to\infty} \max(|a_n|, |b_n|) = 0$ by (4.34). As the $\tilde{\mathcal{L}}^{*n}\mu$ are supported in $[a_n, b_n]$, these measures must converge to δ_0. $\square$

5. Proofs: the self-consistent PFO for the infinite-size system

5.1. Shadowing densities and the asymptotics of $\widetilde{P}$ on $\mathcal{D}$. We are now going to clarify the asymptotics of the self-consistent PFO on the set $\mathcal{D}$ of all densities, proving

Proposition 3 (**Long-term behaviour of $\widetilde{P}$ on $\mathcal{D}$**). *For every $u \in \mathcal{D}$, the sequence $(\widetilde{P}^n u)_{n\geqslant 0}$ converges in $L_1(X, \lambda)$, and*

$$\lim_{n\to\infty} \widetilde{P}^n u \quad \begin{cases} = u_0 & \text{in the stable regime,} \\ \in \{u_{-r_*}, u_0, u_{r_*}\} & \text{in the bistable regime.} \end{cases}$$

The basins $\{u \in \mathcal{D} : \lim_{n\to\infty} \widetilde{P}^n u = u_{\pm r_*}\}$ of the stable fixed points $u_{\pm r_*}$ are L_1-open.

(The set of densities attracted to u_0 in the bistable regime will be discussed in §5.2 below.)

We begin with some notational preparations. Throughout, we fix some $u \in \mathcal{D}$. The iterates $\widetilde{P}^n u$ define parameters $r_n := G(\phi(\widetilde{P}^{n-1}u))$ $(n \geqslant 1)$. With this notation, $\tilde{P}^n u = P_{r_n} \dots P_{r_1} u$.

We let π_N, $N \geqslant 1$, denote the partition of X into monotonicity intervals of $T_{r_N} \circ \dots \circ T_{r_1}$. Note that each branch of this map is a fractional linear bijection from a member of π_N onto X. Since the T_r, $r \in R$, have a common uniform expansion rate, we see that $\operatorname{diam}(\pi_N) \to 0$, and hence, by the standard martingale convergence theorem, $\mathbb{E}[u \parallel \sigma(\pi_N)] \to u$ in $L_1(X, \lambda)$, that is,

$$\eta_N := \|\mathbb{E}[u \parallel \sigma(\pi_N)] - u\|_{L_1(X,\lambda)} \longrightarrow 0 \quad \text{as } N \to \infty. \tag{5.1}$$

Write

$$v_k^{(N)} := P_{r_{N+k}} \dots P_{r_1} \left(\mathbb{E}[u \parallel \sigma(\pi_N)]\right) \quad \text{for } k \geqslant 0 \text{ and } N \geqslant 1,$$

and observe that $v_k^{(N)} \in \mathcal{D}'$ because it is a weighted sum of images of the constant function 1 under various fractional linear branches (recall (4.1) and (4.8)).

For $N = 0$ we let $v_0^{(0)} := u_{r_*}$ and write, in analogy to the notation introduced for $N \geqslant 1$, $v_k^{(0)} := P_{r_k} \dots P_{r_1}(v_0^{(0)})$ and $\eta_0 := \|v_0^{(0)} - u\|_{L_1(X,\lambda)}$. Obviously, $v_k^{(0)} \in \mathcal{D}'$ for all $k \geqslant 0$.

Hence there are measures $\mu_k^{(N)} \in \mathsf{P}(Y)$ such that $v_k^{(N)} = \int_Y w_\bullet \, d\mu_k^{(N)}$. Observe also that

$$\|\widetilde{P}^{N+k}u - v_k^{(N)}\|_{L_1(X,\lambda)} \leqslant \eta_N \quad \text{for all } k \geqslant 0 \text{ and } N \geqslant 0, \tag{5.2}$$

as $\|P_r\| = 1$ for all r, so that in particular

$$|\phi(v_k^{(N)}) - \phi(\widetilde{P}^{N+k}u)| \leqslant \eta_N, \, |G(\phi(v_k^{(N)})) - r_{N+k+1}| \leqslant \left\|G'\right\|_\infty \cdot \eta_N \tag{5.3}$$

In addition, we need to understand the distances

$$\Delta_n^{(N,k)} := \|v_{n+k}^{(N)} - \widetilde{P}^n v_k^{(N)}\|_{L_1(X,\lambda)}.$$

which, in fact, admit some control which is uniform in k:

Lemma 11 (**Shadowing control**). *There is a non-decreasing sequence $(\Delta_n)_{n\geqslant 0}$ in $(0,\infty)$, not depending on $u \in \mathcal{D}$, such that*

$$\Delta_n^{(N,k)} \leqslant \eta_N \cdot \Delta_n \quad \textit{for } k, n \geqslant 0 \textit{ and } N \geqslant 0 \tag{5.4}$$

Proof. Let $r_n^{(N,k)} := G(\phi(\widetilde{P}^{n-1}v_k^{(N)}))$, and observe that (4.17) entails

$$\begin{aligned}\Delta_n^{(N,k)} =& \|P_{r_{N+n+k}} \dots P_{r_{N+1+k}} v_k^{(N)} - P_{r_n^{(N,k)}} \dots P_{r_1^{(N,k)}} v_k^{(N)}\|_{L_1(X,\lambda)} \\ \leqslant& K \cdot \mathsf{d}_W(\mathcal{L}^*_{r_{N+n+k}} \dots \mathcal{L}^*_{r_{N+1+k}} \mu_k^{(N)}, \mathcal{L}^*_{r_n^{(N,k)}} \dots \mathcal{L}^*_{r_1^{(N,k)}} \mu_k^{(N)}).\end{aligned}$$

Applying Lemma 5 repeatedly, we therefore see that

$$\begin{aligned}\Delta_n^{(N,k)} &\leqslant K\,\kappa_2 \sum_{i=0}^{n-1} \kappa_1^i \, |r_{N+n+k-i} - r_{n-i}^{(N,k)}| \\ &= K\,\kappa_2 \sum_{i=0}^{n-1} \kappa_1^i \, |G(\phi(\tilde{P}^{N+n+k-i-1}u)) - G(\phi(\widetilde{P}^{n-i-1}v_k^{(N)}))| \\ &\leqslant K\,\|G'\|_\infty\,\kappa_2 \sum_{i=0}^{n-1} \kappa_1^i \, \|\tilde{P}^{N+n+k-i-1}u - \widetilde{P}^{n-i-1}v_k^{(N)}\|_{L_1(X,\lambda)} \\ &\leqslant K\,\|G'\|_\infty\,\kappa_2 \sum_{i=0}^{n-1} \kappa_1^i \left(\eta_N + \Delta_{n-i-1}^{(N,k)}\right),\end{aligned}$$

where the last inequality uses (5.2). (Recall that $\widetilde{P}$ does not contract on $L_1(X,\lambda)$, whence the need for the $\Delta_{n-i-1}^{(N,k)}$-term.) Letting $K_n := 1 + K\,\|G'\|_\infty\,\kappa_2 \sum_{i=0}^{n-1} \kappa_1^i$ and $\widehat{\Delta}_n^{(N,k)} := \max\{\Delta_i^{(N,k)} : i = 0, \dots, n-1\}$, we thus obtain

$$\widehat{\Delta}_n^{(N,k)} \leqslant K_n \cdot (\eta_N + \widehat{\Delta}_{n-1}^{(N,k)}) \leqslant \dots \leqslant \eta_N \cdot n\,K_n^n, \tag{5.5}$$

which proves our assertion. □

We can now complete the

Proof of Proposition 3. We begin with the easiest situation:

The stable regime. We have to show that $\lim_{n\to\infty} \|\widetilde{P}^n u - u_0\|_{L_1(X,\lambda)} = 0$. Take any $\varepsilon > 0$. Let $(\varepsilon_n)_{n\geqslant 0}$ be the sequence provided by Lemma 9a), and K the constant from (4.17). There is some n (henceforth fixed) for which $K\varepsilon_n < \varepsilon/3$. In view of (5.1), there is some N_0 such that $(1+\Delta_n)\eta_N < 2\varepsilon/3$ whenever $N \geqslant N_0$. We then find, using (5.2), Lemma 11, and (4.17) together with Lemma 9a) that

$$\begin{aligned}\|\widetilde{P}^{N+n}u - u_0\|_{L_1(X,\lambda)} &\leqslant \eta_N + \|v_n^{(N)} - \widetilde{P}^n v_0^{(N)}\|_{L_1(X,\lambda)} + K\varepsilon_n \\ &\leqslant \eta_N + \Delta_n \eta_N + K\varepsilon_n \\ &< \varepsilon \quad \text{for } N \geqslant N_0,\end{aligned} \tag{5.6}$$

which completes the proof in this case.

The bistable regime. Given the sequence $r_n = G(\phi(\widetilde{P}^{n-1}u))$ as before, we let $[a_n, b_n] \subseteq Y$ be the sequence of parameter intervals from Lemma 10. Observe that the measures representing the $v_n^{(N)}$ satisfy $\mathrm{supp}(\mu_n^{(N)}) \subseteq [a_n, b_n]$ for all n and N. We now distinguish two cases:

First case: For all $\varepsilon > 0$ we have ($\clubsuit_\varepsilon$) from Lemma 10. Then, for any $\varepsilon > 0$, the lemma ensures that there is some n (henceforth fixed) with $\max(|a_n|, |b_n|) < \varepsilon/4K$, so that also $\mathsf{d}_W(\mu_n^{(N)}, \delta_0) < \varepsilon/2K$, whatever N. Due to (5.1), $\eta_N < \varepsilon/2$ for $N \geqslant N_0$, and we find, using (5.2) and (4.17),

$$\begin{aligned}\|\widetilde{P}^{N+n}u - u_0\|_{L_1(X,\lambda)} &\leqslant \eta_N + \|v_n^{(N)} - u_0\|_{L_1(X,\lambda)} \\ &\leqslant \eta_N + K\mathsf{d}_W(\mu_n^{(N)}, \delta_0) \\ &< \varepsilon \quad \text{for } N \geqslant N_0,\end{aligned}$$

showing that indeed $\widetilde{P}^n u \to u_0$.

Second case: there is some $\overline{\varepsilon} > 0$ s.t. ($\clubsuit_{\overline{\varepsilon}}$) is violated in that, say,

$$z_{r_n} < a_{n-1} - \overline{\varepsilon} \tag{5.7}$$

for infinitely many n. We show that this implies $\widetilde{P}^n u \to u_{r_*}$. (If ($\clubsuit_{\overline{\varepsilon}}$) is violated in the other direction, $\widetilde{P}^n u \to u_{-r_*}$ then follows by symmetry.)

In view of (5.3), and since (due to $\mu_k^{(N)} \succeq \delta_{a_{N+k}}$, (4.20), and (4.11)) $\phi(v_k^{(N)}) \geqslant \phi(w_{a_{N+k}}) = \bar{w}(a_{N+k})$, we have

$$\begin{aligned}r_{N+k+1} = G(\phi(\widetilde{P}^{N+k}u)) &\geqslant G(\phi(v_k^{(N)}) - \eta_N) \\ &\geqslant G(\phi(v_k^{(N)})) - \left\|G'\right\|_\infty \eta_N \geqslant G(\overline{w}(a_{N+k})) - \left\|G'\right\|_\infty \eta_N,\end{aligned}$$

and hence, observing that $\|\frac{\partial\sigma_r}{\partial r}\|_\infty \leqslant 1$ and writing $\tilde{\sigma}(y) := \sigma_{G(\overline{w}(y))}(y)$ for $y \in Y$,

$$a_{N+k+1} = \sigma_{r_{N+k+1}}(a_{N+k}) \geqslant \tilde{\sigma}(a_{N+k}) - \left\|G'\right\|_\infty \eta_N \tag{5.8}$$

for all N and k. Note that $\tilde{\sigma}'(0) = \sigma_0'(0) + \frac{\partial}{\partial r}\sigma_r(0)|_{r=0} \cdot G'(0) \cdot \overline{w}'(0) = \frac{1}{2} + \frac{1}{2} \cdot G'(0) \cdot \frac{1}{6} > 1$, see (4.28) and (4.10). Therefore, if we fix some $\omega \in (1, \tilde{\sigma}'(0))$, there exists some $a^* > 0$ such that $\tilde{\sigma}(a) \geqslant \omega a$ for all $a \in (0, a^*]$. Without loss of generality, $\overline{\varepsilon}/3 < a^*$.

Now fix N such that $\|G'\|_\infty \eta_N < (\omega - 1)\,\overline{\varepsilon}/3$, and let $N + n + 1$ satisfy (5.7). Due to (4.7), we have

$$a_{N+n+1} = \sigma_{r_{N+n+1}}(a_{N+n}) > \sigma_{r_{N+n+1}}(z_{r_{N+n+1}} + \overline{\varepsilon}) > \overline{\varepsilon}/3. \tag{5.9}$$

Now, if $a_{N+n+1} \geqslant a^*$, then, by (5.8),

$$\begin{aligned} a_{N+n+2} &\geqslant \tilde{\sigma}(a_{N+n+1}) - \left\|G'\right\|_\infty \eta_N \\ &\geqslant \tilde{\sigma}(a^*) - (\omega - 1)\overline{\varepsilon}/3 \\ &\geqslant \omega a^* - (\omega - 1)a^* = a^* > \overline{\varepsilon}/3. \end{aligned}$$

Otherwise, $a_{N+n+1} \in (0, a^*)$, and again

$$\begin{aligned} a_{N+n+2} &\geqslant \tilde{\sigma}(a_{N+n+1}) - \left\|G'\right\|_\infty \eta_N \\ &> \omega\overline{\varepsilon}/3 - (\omega - 1)\,\overline{\varepsilon}/3 = \overline{\varepsilon}/3. \end{aligned}$$

It follows inductively that $\liminf_k a_k \geqslant \overline{\varepsilon}/3$. More precisely: If N_1 and n_1 are integers such that $\eta_{N_1} < \varrho := \|G'\|_\infty^{-1}(\omega - 1)\,\overline{\varepsilon}/3$ and $a_{N_1+n_1} > z_{r_{N_1+n_1+1}} + \overline{\varepsilon}$, then

$$a_k > \overline{\varepsilon}/3 \quad \text{for } k > N_1 + n_1.$$

In particular, if the initial density u is such that $\eta_0 = \|u_{r_*} - u\|_{L_1(X,\lambda)} < \varrho$, then we can take $N_1 = 0$.

Next, fix $y := \overline{\varepsilon}/6 \in (0, \overline{\varepsilon}/3)$, and choose a sequence $(\varepsilon_n)_{n\geqslant 0}$ according to Lemma 9b). Then $0 < y < a_k$ and hence $\delta_0 \prec \delta_y \preceq \mu_k^{(N)}$ for $k > N_1 + n_1$ so that the lemma implies $\mathsf{d}_W(\tilde{\mathcal{L}}^{*n}\mu_k^{(N)}, \mu_{r_*}) \leqslant \varepsilon_n$. Hence, by (4.17),

$$\|\tilde{P}^n v_k^{(N)} - u_{r_*}\|_{L_1(X,\lambda)} \leqslant K \cdot \varepsilon_n \quad \text{for } k > N_1 + n_1 \text{ and all } n, N.$$

We then find, using (5.2) and Lemma 11,

$$\begin{aligned} \|\widetilde{P}^{N+k+n}u - u_{r_*}\|_{L_1(X,\lambda)} &\leqslant \eta_N + \|v_{k+n}^{(N)} - \tilde{P}^n v_k^{(N)}\|_{L_1(X,\lambda)} + K \cdot \varepsilon_n \\ &\leqslant \eta_N + \Delta_n \eta_N + K \cdot \varepsilon_n \end{aligned} \tag{5.10}$$

for $k > N_1 + n_1$ and all n, N. Now $\lim_{n\to\infty} \|\widetilde{P}^n u - u_{r_*}\|_{L_1(X,\lambda)} = 0$ follows as in the stable case.

It remains to prove that the basin of attraction of u_{r_*} is L_1-open. (Then, by symmetry, the same is true for u_{-r_*}.) As $\widetilde{P}$ is L_1-continuous, it suffices to show that this basin contains an open L_1-ball centered at u_{r_*}. To check the latter condition, first notice that $z_{r_*} < 0 < \operatorname{supp}(\mu_{r_*})$ so that there is some $n_1 > 0$ such that $\sigma_{r_*}^{n_1}(a_0) > 0$. As we can assume w.l.o.g. that $\overline{\varepsilon} < |z_{r_*}|$, we have $\sigma_{r_*}^{n_1}(a_0) > z_{r_*} + \overline{\varepsilon}$, and as $\widetilde{P}$ is L_1-continuous, there is some $\overline{\varrho} \in (0, \varrho)$

such that $a_{r_{n_1}} = \sigma_{r_{n_1}} \circ \cdots \circ \sigma_{r_1}(a_0) > z_{r_{n_1}} + \overline{\varepsilon}$ whenever $\|u - u_{r_*}\|_{L_1(X,\lambda)} < \overline{\varrho}$. Therefore we can continue to argue as in the previous paragraph (using the present n_1 and $N_1 = 0$) to conclude that $\lim_{n\to\infty} \|\widetilde{P}^n u - u_{r_*}\|_{L_1(X,\lambda)} = 0$. □

Remark 5. We just proved a bit more than what is claimed in Proposition 3: another look at equation (5.10) reveals that, in the bistable regime, the stable fixed point u_{r_*} of $\widetilde{P}$ is even Lyapunov-stable (and the same is true for u_{-r_*}). Indeed, fix $\overline{\varepsilon} > 0$, $n_1 \in \mathbb{N}$ and $\overline{\varrho} > 0$ as in the preceding paragraph. That choice was completely independent of the particular initial densities investigated there, and the same is true of the choice of the constants K, Δ_n and ε_n occuring in estimate (5.10). Now let $\delta > 0$. Choose $n_2 \in \mathbb{N}$ such that $\varepsilon_{n_2} < \frac{\delta}{2K}$ and then $\eta := \min\{\overline{\varrho}, \frac{\delta}{2(1+\Delta_{n_2})}\}$. Then equation (5.10), applied with $N = 0$, shows that for each $u \in L_1(X, \lambda)$ with $\eta_0 = \|u - u_{r_*}\|_{L_1(X,\lambda)} < \eta$ and for each $n \geqslant 0$ holds

$$\|\widetilde{P}^{n_1+n_2+n} u - u_{r_*}\|_{L_1(X,\lambda)} \leqslant \eta_0(1 + \Delta_{n_2}) + K\,\varepsilon_{n_2} < \delta\ .$$

5.2. **The stable manifold of u_0 in the bistable regime.** Let $W^s(u_0) := \{u \in \mathcal{D} : \widetilde{P}^n u \to u_0\}$ denote the *stable manifold* of u_0 in the space of *all* probability densities on X. Clearly, all symmetric densities u (i.e. those satisfying $u(-x) = u(x)$) belong to $W^s(u_0)$, because symmetric densities have field $\phi(u) = 0$ so that also the parameter $G(\phi(u)) = 0$, and symmetry is preserved under the operator P_0.

However, $W^s(u_0)$ is not a big set. In the present section we prove

Proposition 4 (**The basins of $u_{\pm r_*}$ touch $W^s(u_0) \cap \mathcal{D}'$**). *Each density in $W^s(u_0) \cap \mathcal{D}'$ belongs to the boundaries of the basins of u_{r_*} and of u_{-r_*}.*

We start by providing more information on the fields $\phi(\widetilde{P}^n u)$ of orbits in $W^s(u_0) \cap \mathcal{D}'$. Recall that for $u = \int_Y w_\bullet\, d\mu \in \mathcal{D}'$ we have $\widetilde{P}^n u = \int_Y w_\bullet\, d(\tilde{\mathcal{L}}^{*n}\mu)$ ($n \geqslant 0$). Given such a density, we denote by $R_n(u)$ the "radius" of the support of $\tilde{\mathcal{L}}^{*n}\mu$, i.e. $R_n(u) := \inf\{\varepsilon > 0 : \mathrm{supp}(\tilde{\mathcal{L}}^{*n}\mu) \subseteq [-\varepsilon, \varepsilon]\}$, and let $\phi_n(u) := \phi(\widetilde{P}^n u) = \int_Y \overline{w}\, d(\tilde{\mathcal{L}}^{*n}\mu)$.

Lemma 12 (**Field versus support radius**). *In the bistable regime, for each $u \in W^s(u_0) \cap \mathcal{D}'$ there exists a constant $C_u > 0$ such that*

$$|\phi_n(u)| \leqslant C_u \cdot (R_n(u))^2 \quad \textit{for} \ \ n \geqslant 0. \tag{5.11}$$

Proof. In view of the explicit formula (4.10), we have $\overline{w}'(0) = \frac{1}{6}$ and $\overline{w}''(0) = 0$, and therefore see that there is some $\overline{\varepsilon} \in (0, \frac{1}{3})$ such that for every $\varepsilon \in (0, \overline{\varepsilon})$ and all $y \in [-2\varepsilon, 2\varepsilon]$,

$$\frac{|y|}{6} \leqslant |\overline{w}(y)| \leqslant \frac{|y|}{6 - 6\varepsilon^2} \qquad \text{and} \qquad |G(y)| > (B - c\varepsilon)|y|, \tag{5.12}$$

where $B := G'(0) > 6$ and c, too, is a positive constant which only depends on the function G. In addition, elementary calculations based on (4.2) and

(4.6) show that letting $\kappa := \max(1, \frac{B+2}{6})$, $\overline{\varepsilon}$ can be chosen such that, for every $\varepsilon \in (0, \overline{\varepsilon})$ and $r \in [0, B\varepsilon)$, also

$$|\sigma_r'(y) - \frac{1}{2}| \leqslant B\varepsilon \ , \quad |\tau_r'(y) - \frac{1}{2}| \leqslant B\varepsilon \quad \text{for } |y| \leqslant \varepsilon,$$

$$B\varepsilon \geqslant \tau_r(y) \geqslant \sigma_r(y) \geqslant \left(\frac{1}{2} - \kappa\varepsilon\right)(y+r) \geqslant 0 \quad \text{for } y \in [-r, \varepsilon], \text{ and} \tag{5.13}$$

$$0 > \tau_r(y) \geqslant \sigma_r(y) \geqslant \left(\frac{1}{2} + \varepsilon\right)(y+r) > -B\varepsilon \quad \text{for } y \in [-\varepsilon, -r).$$

(Recall that σ_r and τ_r share a zero at $z_r = -r$.) Finally, note that we can w.l.o.g. take $\overline{\varepsilon}$ so small that $\bar{B} := \left(\frac{1}{2} - \overline{\varepsilon}\right)\left(1 + \frac{B}{6} - (\frac{1}{3} + \frac{c}{6})\overline{\varepsilon}\right) \in (1, 3]$. (Due to Assumption I we have $B \leqslant 25$.)

Consider some $v = \int_Y w_\bullet \, d\nu$ with $\nu \in \mathsf{P}(Y)$. We claim that for $\varepsilon \in (0, \frac{\overline{\varepsilon}}{\kappa})$

$$|\phi(\widetilde{P}v)| \geqslant \bar{B} \cdot |\phi(v)| - \varepsilon^2 \quad \text{if } \operatorname{supp}(\nu) \subseteq [-\varepsilon, \varepsilon]. \tag{5.14}$$

Denote $r := G(\phi(v))$ which by S-shapedness of G satisfies $|r| < B\varepsilon$. In view of our system's symmetry, we may assume w.l.o.g. that $r \geqslant 0$. According to (4.11) and (4.13) we have

$$\phi(\widetilde{P}v) = \phi\left(\int_Y w_\bullet \, d(\widetilde{\mathcal{L}}^*\nu)\right) = \int_Y (\overline{w} \circ \sigma_r) \cdot p_r \, d\nu + \int_Y (\overline{w} \circ \tau_r) \cdot (1 - p_r) \, d\nu$$

so that, due to (5.12) and (5.13),

$$\begin{aligned}
&\int_Y (\overline{w} \circ \sigma_r) \cdot p_r \, d\nu \\
&\quad \geqslant \int_Y \left(\frac{1_{[-B\varepsilon, 0)} \circ \sigma_r(y)}{6 - 6\varepsilon^2} + \frac{1_{[0, B\varepsilon]} \circ \sigma_r(y)}{6}\right) \cdot \sigma_r(y) \, p_r(y) \, d\nu(y) \\
&\quad \geqslant \int_Y \left(\frac{1_{[-\frac{2}{3}, -r)}(y)}{6 - 6\varepsilon^2}\left(\frac{1}{2} + \varepsilon\right) + \frac{1_{[-r, \frac{2}{3}]}(y)}{6}\left(\frac{1}{2} - \kappa\varepsilon\right)\right) \cdot (y + r) \, p_r(y) \, d\nu(y).
\end{aligned}$$

Combining this with the parallel estimate for $\int_Y (\overline{w} \circ \tau_r) \cdot (1 - p_r) \, d\nu$, we get

$$\phi(\widetilde{P}v) \geqslant \int_{[-\frac{2}{3}, -r)} \frac{\left(\frac{1}{2} + \varepsilon\right)(y + r)}{6 - 6\varepsilon^2} \, d\nu(y) + \int_{[-r, \frac{2}{3}]} \frac{\left(\frac{1}{2} - \kappa\varepsilon\right)(y + r)}{6} \, d\nu(y).$$

Continuing, we find that

$$\begin{aligned}
\phi(\widetilde{P}v) &\geqslant \int_{[-\frac{2}{3}, -r)} \frac{\frac{1}{2} + \varepsilon}{6 - 6\varepsilon^2} \cdot y \, d\nu(y) + \int_{[-r, \frac{2}{3}]} \frac{\frac{1}{2} - \kappa\varepsilon}{6} \cdot y \, d\nu(y) + \frac{\frac{1}{2} - \kappa\varepsilon}{6} \cdot r \\
&\geqslant \int_{[-\frac{2}{3}, 0)} \frac{\frac{1}{2} + \varepsilon}{6 - 6\varepsilon^2} \cdot y \, d\nu(y) + \int_{[0, \frac{2}{3}]} \frac{\frac{1}{2} - \kappa\varepsilon}{6} \cdot y \, d\nu(y) + \frac{\frac{1}{2} - \kappa\varepsilon}{6} \cdot r \\
&\geqslant \int_{[-\frac{2}{3}, 0)} K \cdot \overline{w}(y) \, d\nu(y) + \int_{[0, \frac{2}{3}]} K^* \cdot \overline{w}(y) \, d\nu(y) + \frac{\frac{1}{2} - \kappa\varepsilon}{6} \cdot r \ ,
\end{aligned}$$

where $K := (\frac{1}{2}+\varepsilon)/(1-\varepsilon^2) > K^* := (\frac{1}{2}-\kappa\varepsilon)(1-\varepsilon^2)$. As, because of (5.12), $\phi(v) = \int_Y \overline{w}\, d\nu \leqslant \frac{\varepsilon}{5}$, so that $r = G(\phi(v)) \geqslant (B-c\varepsilon)\cdot\phi(v)$, we conclude

$$\begin{aligned}\phi(\widetilde{P}v) &\geqslant \phi(v)\left(K^* + \frac{(\frac{1}{2}-\kappa\varepsilon)(B-c\varepsilon)}{6}\right) + (K-K^*)\int_{[-\frac{2}{3},0)} \overline{w}(y)\, d\nu(y)\\ &\geqslant \phi(v)\left(\frac{1}{2}-\overline{\varepsilon}\right)\left(1+\frac{B}{6}-\left(\frac{1}{3}+\frac{c}{6}\right)\overline{\varepsilon}\right) - \varepsilon^2 = \bar{B}\cdot\phi(v) - \varepsilon^2,\end{aligned}$$

since $K-K^* \leqslant 3\varepsilon$ and $|\overline{w}(y)| \leqslant \frac{\varepsilon}{3}$ whenever $|y| \leqslant \varepsilon \leqslant \frac{1}{3}$. This proves (5.14).

Now take any $u \in W^s(u_0)\cap\mathcal{D}'$. Then $\phi_n(u) \to 0$, and the second alternative of Proposition 2 applies, so that $R_n(u) \leqslant \overline{\varepsilon}/\kappa$ and $(1+2BR_n(u))^2 \leqslant \frac{\bar{B}+1}{2}$ for all n larger than some $n_{\overline{\varepsilon}}$. In particular,

$$R_{n+1}(u)^2 \leqslant (1+2BR_n(u))^2 R_n(u)^2 \leqslant \frac{\bar{B}+1}{2} R_n(u)^2 \tag{5.15}$$

for these n in view of (5.13). Applying, for $n \geqslant n_{\overline{\varepsilon}}$, the estimate (5.14) to $v := \widetilde{P}^n u$ and $\varepsilon := R_n(u)$, we obtain

$$|\phi_{n+1}(u)| \geqslant \bar{B}\cdot|\phi_n(u)| - (R_n(u))^2 \quad \text{for } n \geqslant n_{\overline{\varepsilon}}.$$

Suppose for a contradiction that $(R_n(u))^2 < \frac{\bar{B}-1}{2}\,|\phi_n(u)|$ for some $n > n_{\overline{\varepsilon}}$. Then $|\phi_{n+1}(u)| > \frac{\bar{B}+1}{2}|\phi_n(u)|$, and therefore $(R_{n+1}(u))^2 \leqslant \frac{\bar{B}+1}{2}(R_n(u))^2 < \frac{\bar{B}-1}{2}\frac{\bar{B}+1}{2}|\phi_n(u)| < \frac{\bar{B}-1}{2}|\phi_{n+1}(u)|$. We can thus continue inductively to see that $|\phi_n(u)| < |\phi_{n+1}(u)| < |\phi_{n+2}(u)| < \dots$ which contradicts $\phi_n(u) \to 0$. Therefore $|\phi_n(u)| \leqslant \frac{2}{\bar{B}-1}\,(R_n(u))^2$ for all $n > n_{\overline{\varepsilon}}$, and the assertion of our lemma follows. □

Lemma 13 ($W^s(u_0)$ **is a thin set for the order** $\prec$). *In the bistable regime, if $u = \int_Y w_\bullet\, d\mu$ and $v = \int_Y w_\bullet\, d\nu$ are densities in $\mathcal{D}'$ with $\mu \prec \nu$, then at most one of u and v can belong to $W^s(u_0)$.*

Proof. Suppose that $u \in W^s(u_0)$. We are going to show that $\widetilde{P}^n v \to u_{r_*}$, i.e. $\tilde{\mathcal{L}}^{*n}\nu \to \mu_{r_*}$ as $n \to \infty$.

Assume for a contradiction that also $v \in W^s(u_0)$. We denote the parameters obtained from u by $r_{n,\mu} := G(\phi(\tilde{P}^{n-1}u)) = G(\int_Y \overline{w}\, d(\tilde{\mathcal{L}}^{*(n-1)}\mu))$, and define $r_{n,\nu}$ analogously. Then our assumption implies that $\lim_{n\to\infty} r_{n,\mu} = \lim_{n\to\infty} r_{n,\nu} = 0$.

In view of (4.10), $\overline{w}' \geqslant \frac{1}{6}$, and one checks immediately that $\inf_Y \sigma_0' = \frac{18}{49} > \frac{1}{3}$ so that there is $n_0 > 0$ such that $\inf_Y \sigma'_{r_{n,\mu}} \geqslant \frac{1}{3}$ for all $n \geqslant n_0$. Because of the strict monotonicity of $\tilde{\mathcal{L}}^*$ (Lemma 6) we have $\tilde{\mathcal{L}}^{*n_0}\mu \prec \tilde{\mathcal{L}}^{*n_0}\nu$, so that (replacing μ and ν by these iterates) we can assume w.l.o.g. that $n_0 = 0$. Denote $\mathcal{L}_\mu^{*(n)} := \mathcal{L}^*_{r_{n,\mu}} \circ \dots \circ \mathcal{L}^*_{r_{1,\mu}}$ so that $\tilde{\mathcal{L}}^{*n}\mu = \mathcal{L}_\mu^{*(n)}\mu$ and ($\mu \mapsto r_\mu$ being non-decreasing) $\tilde{\mathcal{L}}^{*n}\nu \succeq \mathcal{L}_\mu^{*(n)}\nu$ for $n \geqslant 1$. Therefore

$$\begin{aligned} r_{n,\nu} - r_{n,\mu} &\geqslant G\left(\int_Y \overline{w}\, d(\mathcal{L}_\mu^{*(n)}\nu)\right) - G\left(\int_Y \overline{w}\, d(\mathcal{L}_\mu^{*(n)}\mu)\right) \\ &\geqslant \inf_X G' \cdot \left(\int_Y \overline{w}\, d(\mathcal{L}_\mu^{*(n)}\nu) - \int_Y \overline{w}\, d(\mathcal{L}_\mu^{*(n)}\mu)\right). \end{aligned}$$

In view of the lower bounds for $\overline{w}'$ and $\sigma'_{r_{n,\mu}}, \tau'_{r_{n,\mu}}$, repeated application of the estimate (4.30) from Lemma 7 yields

$$r_{n,\nu} - r_{n,\mu} \geqslant \frac{\inf_X G'}{6 \cdot 3^n} \int_Y \mathrm{id}\; d(\nu - \mu). \tag{5.16}$$

Observe that the last integral is strictly positive because $\mu \prec \nu$, cf. (4.22).

On the other hand, due to Proposition 2 there are $\varepsilon_n \searrow 0$ such that

$$\operatorname{supp}(\tilde{\mathcal{L}}^{*n}\mu) \cup \operatorname{supp}(\tilde{\mathcal{L}}^{*n}\nu) \subseteq [-\varepsilon_n, \varepsilon_n],$$

and as $\sigma_0'(0) = \frac{1}{2} < \frac{5}{9}$ and $r_{n,\mu}, r_{n,\nu} \to 0$ (whence also $z_{r_{n,\mu}}, z_{r_{n,\nu}} \to z_0 = 0$), there exists a constant $C > 0$ such that $\varepsilon_n \leqslant C(\frac{5}{9})^n$ for $n \geqslant n'$. Hence $|\phi_n(u)|, |\phi_n(v)| \leqslant \max\{C_u, C_v\} \cdot C^2(\frac{25}{81})^n$ for $n \geqslant n'$ by Lemma 12, and as $r_{n,\nu} - r_{n,\mu} \leqslant \sup \overline{w}' \cdot (|\phi_n(u)| + |\phi_n(v)|)$, this contradicts the previous estimate (5.16). □

We can now conclude this section with the

Proof of Proposition 4. Suppose that $u = \int w_\bullet\, d\mu \in W^s(u_0)$. For $t \in (0,1)$ let $u^{(t)} := \int_Y w_\bullet\, d((1-t)\mu + t\delta_{2/3}) \in \mathcal{D}'$. Then $u^{(t)} \succ u$, hence $u^{(t)} \notin W^s(u_0)$ by the previous proposition. Therefore, due to Proposition 2 and monotonicity of $\tilde{\mathcal{L}}^*$, for any t, $\widetilde{P}^n u^{(t)}$ converges to $u_{r_*} \succ u_0$ as $n \to \infty$.

On the other hand, $\lim_{t\to 0} \|u - u^{(t)}\|_{L_1(X,\lambda)} = 0$, so u is in the boundary of the basin of u_{r_*}. Replacing $\delta_{2/3}$ by $\delta_{-2/3}$ yields the corresponding result for the basin of u_{-r_*}. □

5.3. **Differentiability of $\widetilde{P}$ at $\mathcal{C}^2$-densities.** As $\widetilde{P}$ is based on a parametrised family of PFOs where the *branches* of the underlying map (and not only their weights) depend on the parameter, it is nowhere differentiable, neither as an operator on $L_1(X,\lambda)$ nor as an operator on the space $\mathrm{BV}(X)$ of (much more regular) functions of bounded variation on X. On the other hand, as the branches of the map and their parametric dependence are analytic, one can show that $\widetilde{P}$ is differentiable as an operator on the space of functions that can be extended holomorphically to some complex neighbourhood of $X \subseteq \mathbb{C}$.

Here we will focus on a more general but slightly weaker differentiability statement.

Lemma 14 (**Differentiability of $\widetilde{P}$ at C^2-densities**). *Let $u \in \mathcal{C}^2(X)$ be a probability density w.r.t. λ and let $g \in L_1(X,\lambda)$ have $\int_X g\, d\lambda = 0$. Then*

$$\frac{\partial}{\partial\tau}\widetilde{P}(u+\tau g)|_{\tau=0} = P_r(g) + w_r(u) \cdot G'(\phi(u))\, \phi(g) \tag{5.17}$$

where $r = G(\phi(u))$, $w_r(u) := P_r\left((u\,v_r)'\right)$, *and* $v_r(x) = \frac{4x^2-1}{4-r^2}$. *If we consider* $\widetilde{P}$ *as an operator from* $\mathrm{BV}(X)$ *to* $L_1(X,\lambda)$, *then* $\widetilde{P}$ *is even differentiable at each probability density* $u \in \mathcal{C}^2(X) \subset \mathrm{BV}(X)$ *and*

$$D\widetilde{P}|_u = P_r + G'(\phi(u))\,w_r(u) \otimes \phi\ . \tag{5.18}$$

Proof. In order to simplify the notation define a kind of transfer operator L by $Lu := u + u \circ f_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$ and note that $(Lu)' = Lu'$. Observing that $f_{N_r^{-1}} = f_{M_r^{-1}} \circ f_{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}$, we have $P_r u = L(u \circ f_{M_r^{-1}} \cdot f'_{M_r^{-1}})$. Define

$$v_r(x) := \left(\frac{\partial}{\partial r} f_{M_r^{-1}}\right)(f_{M_r}(x)) = \frac{4x^2-1}{4-r^2}. \tag{5.19}$$

For a function $u \in \mathcal{C}^2(X)$ denote by U the antiderivative of u. Then

$$\begin{aligned} u \circ f_{M_s^{-1}} \cdot f'_{M_s^{-1}} - u \circ f_{M_r^{-1}} \cdot f'_{M_r^{-1}} &= \Big(U \circ f_{M_s^{-1}} - U \circ f_{M_r^{-1}}\Big)' \\ &= \left((s-r)\cdot \frac{\partial}{\partial r}(U \circ f_{M_r^{-1}}) + R_{s,r}\right)' \end{aligned} \tag{5.20}$$

where

$$R_{s,r}(x) := \int_r^s (s-t)\,\frac{\partial^2}{\partial t^2}(U(f_{M_t^{-1}}(x)))\,dt\ .$$

As $\frac{\partial}{\partial r}(U \circ f_{M_r^{-1}}) = u \circ f_{M_r^{-1}} \cdot \frac{\partial}{\partial r} f_{M_r^{-1}} = (u\,v_r) \circ f_{M_r^{-1}}$, we have

$$\left(\frac{\partial}{\partial r}(U \circ f_{M_r^{-1}})\right)' = (u\,v_r)' \circ f_{M_r^{-1}} \cdot f'_{M_r^{-1}}\ .$$

Together with (5.20) this yields

$$\begin{aligned} P_s u - P_r u &= L\left(u \circ f_{M_s^{-1}} \cdot f'_{M_s^{-1}} - u \circ f_{M_r^{-1}} \cdot f'_{M_r^{-1}}\right) \\ &= (s-r)\,L\left((u\,v_r)' \circ f_{M_r^{-1}} \cdot f'_{M_r^{-1}}\right) + L\left(R'_{s,r}\right) \\ &= (s-r)\,P_r\left((u\,v_r)'\right) + L\left(R'_{s,r}\right) \end{aligned}$$

and $|L(R'_{s,r})(x)| \leqslant C\,(s-r)^2$ with a constant that involves only the first two derivatives of u.

Now let $u \in \mathcal{C}^2(X)$ be a probability density, and let $g \in L_1(X,\lambda)$ be such that $\int g\,d\lambda = 0$. Let $r := G(\phi(u))$ and $s := G(\phi(u+g))$. Then

$$\begin{aligned} \widetilde{P}(u+g) - \widetilde{P}(u) &= (P_s u - P_r u) + P_r g + (P_s g - P_r g) \\ &= (s-r)\,P_r\left((u\,v_r)'\right) + P_r(g) + (P_s g - P_r g) + L(R'_{s,r})\ . \end{aligned} \tag{5.21}$$

This implies at once formula (5.17) for the directional derivative, and as $\|P_s g - P_r g\|_1 \to 0$ $(s \to r)$ uniformly for g in the unit ball of $\mathrm{BV}(X)$, also (5.18) follows at once. $\square$

Proposition 5 ($u \equiv 1$ **is a hyperbolic fixed point of** $\widetilde{P}$). *In the bistable regime, $u \equiv 1$ is a hyperbolic fixed point of $\widetilde{P}|_{\mathcal{D}\cap \mathrm{BV}(X)}$ in the following sense: the derivative of $\widetilde{P} : \mathcal{D}\cap \mathrm{BV}(X) \to L_1(X, \lambda)$ at $u \equiv 1$ has a one-dimensional unstable subspace and a codimension 1 stable subspace.*

Proof. Let $Q := D\widetilde{P}|_{u\equiv 1}$. As $G'(0) = B$ and $w_0(1) = P_0[2x] = [x]$, it follows from (5.18) that $Q = P_0 + B\,[x]\otimes\phi$. (Here $[2x]$ denotes the function $x \mapsto 2x$, etc.) Observe now that $\phi([x]) = \frac{1}{12}$. Then $Q[x] = P_0[x] + \frac{B}{12}\,[x] = (\frac{1}{2} + \frac{B}{12})[x]$ so that, for $B > 6$, Q has the unstable eigendirection $[x]$ with eigenvalue $\lambda := \frac{1}{2} + \frac{B}{12} > 1$. On the other hand, as $\phi(1) = 0$, we have $Q1 = P_0 1 = 1$, so the constant density 1 is a neutral eigendirection, and finally, for $f \in \ker(\phi) \cap \ker(\lambda)$, we have $Qf = P_0 f$, so $\mathrm{Var}(Qf) \leqslant \frac{1}{2}\,\mathrm{Var}(f)$. □

6. The noisy system

In Theorem 3 we proved that, in the bistable regime, each weak accumulation point of the sequence $(\boldsymbol{\mu}_N \circ \epsilon_N^{-1})_{N\geqslant 1}$ is of the form $\alpha\,\delta_{u_{-r_*}\lambda} + (1 - 2\alpha)\,\delta_{u_0\lambda} + \alpha\,\delta_{u_{r_*}\lambda}$ for some $\alpha \in [0, \frac{1}{2}]$, i.e. that the stationary states of the finite-size systems approach a mixture of the stationary states of the infinite-size system. It is natural to expect that actually $\alpha = \frac{1}{2}$, meaning that any limit state thus obtained is a mixture of *stable* stationary states of $\widetilde{P}$. While we could not prove this for the model discussed so far, we now argue that this conjecture can be verified if we add some noise to the systems.

At each step of the dynamics we perturb the parameter of the single-site maps by a small amount. To make this idea more precise, let

$$
\begin{aligned}
r(Q,t) &= G(\phi(Q) + t) \quad \text{for } Q \in \mathsf{P}(X) \text{ and } t \in \mathbb{R}\,, \text{ in particular} \\
r(\mathbf{x},t) &= G(\phi(\mathbf{x}) + t) \quad \text{for } \mathbf{x} \in X^N \text{ and } t \in \mathbb{R}\,.
\end{aligned} \tag{6.1}
$$

Let $\eta_1, \eta_2, \ldots$ be i.i.d. symmetric real valued random variables with common distribution ϱ and $|\eta_n| \leqslant \varepsilon$. For $n = 1, 2, \ldots$ and $\mathbf{x} \in X^N$ let us define the X^N-valued Markov process $(\xi_n)_{n\in\mathbb{N}}$ by $\xi_0 = \mathbf{x}$ and

$$
\xi_{n+1} = T_{r(\xi_n,\eta_{n+1})}(\xi_n). \tag{6.2}
$$

Assume now that the distribution of ξ_n has density h_n w.r.t. Lebesgue measure on X^N. Then routine calculations show that the distribution of ξ_{n+1} has density $\int_{\mathbb{R}} P_{N,t} h_n \, d\varrho(t)$ where $P_{N,t}$ is the PFO of the map $\mathbf{T}_{N,t} : X^N \to X^N$, $(\mathbf{T}_{N,t}(\mathbf{x}))_i = T_{r(\mathbf{x},t)}(x_i)$. It is straightforward to check that, for sufficiently small ε, Lemmas 2 – 4 from Section 3 carry over to all $\mathbf{T}_{N,t}$ ($|t| \leqslant \varepsilon$) with uniform bounds, and that $\int_{X^N} |P_{N,t}f - P_{N,0}f|\,d\lambda^N \leqslant \mathrm{const}_N \cdot \varepsilon \cdot \mathrm{Var}(f)$ so that the perturbation theorem of [11] guarantees that the process $(\xi_n)_{n\in\mathbb{N}}$ has a unique stationary probability $\boldsymbol{\mu}_{N,\varepsilon}$ whose density w.r.t. λ^N tends, in $L_1(X^N, \lambda^N)$, to the unique invariant density of $\mathbf{T}_N$ as $\varepsilon \to 0$. This convergence is not uniform in N, however. Nevertheless, folklore arguments show that there is some $\tilde{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \tilde{\varepsilon})$ and all $N \in \mathbb{N}$ the absolutely continuous stationary measure $\boldsymbol{\mu}_{N,\varepsilon}$ is unique so that the symmetry properties of the maps T_r and the random variables η_n guarantee that $\boldsymbol{\mu}_{N,\varepsilon}$ is symmetric in the sense that its density $h_{N,\varepsilon}$ satisfies $h_{N,\varepsilon}(x) = h_{N,\varepsilon}(-x)$.

On the other hand, for each fixed $\varepsilon > 0$, all weak limit points of the measures $\boldsymbol{\mu}_{N,\varepsilon} \circ \epsilon_N^{-1}$ as $N \to \infty$ are stationary probabilities for the $\mathsf{P}(X)$-valued Markov process $(\Xi_n)_{n\in\mathbb{N}}$ defined by

$$\Xi_{n+1} = \Xi_n \circ T^{-1}_{r(\Xi_n,\eta_{n+1})} , \tag{6.3}$$

compare the definition of $\widetilde{T} : \mathsf{P}(X) \to \mathsf{P}(X)$ in (2.9). The proof is completely analogous to the corresponding one for the unperturbed case (see Lemma 1 and Corollary 1). For $\varepsilon \in (0, \tilde{\varepsilon})$ the symmetry of the $\boldsymbol{\mu}_{N,\varepsilon}$ carries over to these limit measures Q in the sense that $Q(A) = Q\{\hat{\mu} : \mu \in A\}$ for each Borel measurable set $A \subseteq \mathsf{P}(X)$ where $\hat{\mu}(U) := \mu(-U)$ for all Borel subsets $U \subseteq X$.

The following proposition then shows that, in the bistable regime and for small $\varepsilon > 0$ and large N, the measures $\boldsymbol{\mu}_{N,\varepsilon}$ are weakly close to the mixture $\frac{1}{2}\left((u_{-r_*}\lambda)^{\mathbb{N}} + (u_{r_*}\lambda)^{\mathbb{N}}\right)$ of the stable states for $\widetilde{P}$; compare also Theorem 3.

Proposition 6 (**Invariant measures for infinite-size noisy systems**). *Suppose $G'(0) > 6$ so that we are in the bistable regime and recall that the η_n are symmetric random variables.*

Then, for every $\delta > 0$ there is $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the stationary distribution Q_ε of Ξ_n on $\mathsf{P}(X)$ is supported on the set of measures $u \cdot \lambda \in \mathsf{P}(X)$ which have density $u = \int_Y w_\bullet \, d\mu \in \mathcal{D}'$ with representing measures $\mu \in \mathsf{P}(Y)$ satisfying $\mathsf{d}_W(\mu, \frac{1}{2}(\mu_{-r_*} + \mu_{r_*})) \leqslant \delta$.

Sketch of the proof. Let Q be a stationary distribution of Ξ_n that occurs as a weak limit of the measures $\boldsymbol{\mu}_{N,\varepsilon}$. So Q is symmetric. Just as in the proof of Theorem 3, where the "zero noise limit", namely the transformation $\widetilde{T}$ is treated, one argues that Q is supported by the set of measures $u \cdot \lambda$, $u \in \mathcal{D}$. Arguing as in the derivation of (5.2) one shows that densities in the support of Q can be approximated in $L_1(X, \lambda)$ by densities from $\mathcal{D}'$, and the stationarity of Q implies that Q is indeed supported by measures with densities from $\mathcal{D}'$. Therefore the process $(\Xi_n)_{n\geqslant 0}$ can be described by the transfer operator $\widetilde{\mathcal{L}}^*_\varepsilon$ of an iterated function system on Y just as the self-consistent PFO $\widetilde{P}$ is described by the operator $\widetilde{\mathcal{L}}^*$ in equation (4.13). The only difference is that in this case one first chooses the parameter r randomly, $r = G(\int_Y \overline{w} \, d\mu + \eta_{n+1})$ and then the branch σ_r or τ_r with respective probabilities $p_r(y)$ and $(1 - p_r(y))$.

Let $y > 0$ be such that $\mathbb{P}\{\eta_n > y\} > 0$. Suppose now that for some realisation of the process $(\Xi_n)_{n\geqslant 0}$ the numbers $r(\Xi_n, \eta_{n+1})$ satisfy condition $(\clubsuit_\varepsilon)$ of Lemma 10 for all $\varepsilon > 0$. Then it follows, as in the proof of Proposition 2, that $\lim_{n\to\infty} r_n(\Xi_n, \eta_{n+1}) = 0$ and the measures Ξ_n converge weakly to λ so that also $\lim_{n\to\infty} r(\Xi_n, 0) = 0$. As $\eta_n > y > 0$ for infinitely many n almost surely, both limit cannot be zero at the same time, and we conclude that almost surely there is some $\varepsilon > 0$ such that ($\clubsuit_\varepsilon$) is not satisfied. In particular, there are $\bar\varepsilon > 0$ and $\bar n \in \mathbb{N}$ such that ($\clubsuit_{\bar\varepsilon}$) is violated for $n = \bar n - 1$ with some positive probability κ.

Let $\Xi_n = h_n \cdot \lambda$ with $h_n = \int_Y w_\bullet \, d\nu_n$. (So h_n and ν_n are random objects.) As in (5.9) we conclude that $\sup \operatorname{supp}(\nu_{\bar n}) < -\bar\varepsilon/3$ or $\inf \operatorname{supp}(\nu_{\bar n}) > \bar\varepsilon/3$ in this case. Without loss of generality we assume that the latter happens with probability at least $\frac{\kappa}{2}$.

Next, as in (4.32) we may choose $y \in (0, \bar\varepsilon/3)$ so small that $0 < y < y_1 := \sigma_{G(\overline{w}(y))}(y) \leqslant \inf \operatorname{supp}(\tilde{\mathcal{L}}^* \delta_y)$. Hence, for reasons of continuity, there is $\varepsilon_1 > 0$ such that also $y \leqslant \inf \operatorname{supp}(\tilde{\mathcal{L}}_\varepsilon^* \delta_y)$ if $\varepsilon \in [0, \varepsilon_1)$. Therefore, in view of the monotonicity of the operator $\tilde{\mathcal{L}}_\varepsilon^*$, we can conclude that $\inf \operatorname{supp}(\nu_n) \geqslant y$ for all $n \geqslant \bar n$ with probability at least $\frac{\kappa}{2}$. Now fix $\delta > 0$. By Lemma 9b there is some (non-random) $n_1 \in \mathbb{N}$ such that $\mathsf{d}_W(\tilde{\mathcal{L}}^{*n_1} \nu_n, \mu_{r_*}) \leqslant \frac{\delta}{2}$ for all $n \geqslant \bar n$ with probability at least $\frac{\kappa}{2}$. But then, by continuity reasons again, there is $\varepsilon_0 \in (0, \varepsilon_1)$ such that $\mathsf{d}_W(\nu_{n+n_1}, \mu_{r_*}) < \delta$ for all $n \geqslant \bar n$ with probability at least $\frac{\kappa}{2}$. The claim of the proposition follows now, because $(\Xi_n)_n$ is a Markov process and because the stationary distribution Q is symmetric. □

Appendix A. Some technical and numerical results

A.1. **Proof of Lemma 1.** It suffices to prove the convergence for evaluations of any Lipschitz continuous function φ defined on X. Let us denote $r_n = r(Q_n)$ (resp. $r = r(Q)$), and α_n (resp. α) the discontinuity point of T_{r_n} (resp. T_r). Recall that $\alpha_n = -\frac{r_n}{4}$ (resp. $\alpha = -\frac{r}{4}$).

Let us fix $\varepsilon > 0$. Q being non-atomic, there exists $\delta > 0$ such that the interval $U := [c-\delta, c+\delta]$ is of Q-measure smaller that ε. The weak convergence of Q_n to Q implies that c_n tends to c, and that $\limsup_{n\to+\infty} Q_n(U) \leqslant Q(U)$. Let us choose n_0 such that for all $n \geqslant n_0$, $|c_n - c| < \frac{\delta}{2}$ and $Q_n(U) < \varepsilon$. One then has

$$
\begin{aligned}
\Big| \int_X \varphi \, d(\widetilde{T} Q) - \int_X \varphi \, d(\widetilde{T} Q_n) \Big| &= \Big| \int_X \varphi \circ T_r \, dQ - \int_X \varphi \circ T_{r_n} \, dQ_n \Big| \\
&\leqslant \Big| \int_X \varphi \circ T_r \, d(Q - Q_n) \Big| + \Big| \int_{U^c} (\varphi \circ T_{r_n} - \varphi \circ T_r) \, dQ_n \Big| \\
&\qquad + \Big| \int_U (\varphi \circ T_{r_n} - \varphi \circ T_r) \, dQ_n \Big| \\
&\leqslant \Big| \int_X \varphi \circ T_r \, d(Q - Q_n) \Big| + \operatorname{Lip}(\varphi) \sup_{U^c} |T_{r_n} - T_r| + 2\varepsilon \|\varphi\|_\infty
\end{aligned} \tag{A.1}
$$

Since the application $\varphi \circ T_r$ as a single discontinuity point, which is of zero Q-measure, the first term converges to zero. The second one also goes to zero since it measures the dependence of T_r on its parameter away from the discontinuity point (one can make an explicit computation).

A.2. **The fields of the densities u_r.** We start with some observations on the function $\psi(r) := \phi(u_r)$ that are based on symbolic computations and on

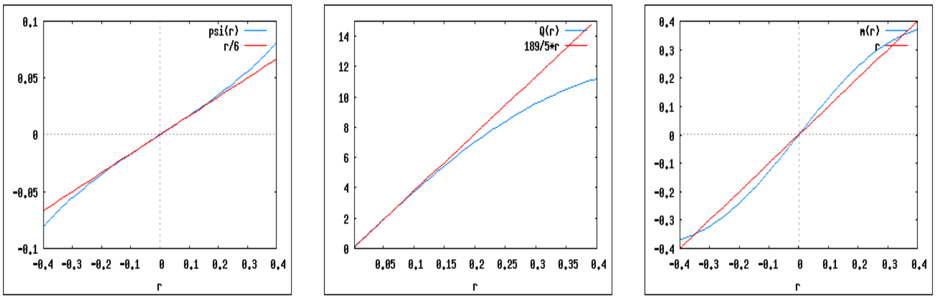

FIGURE 3. The functions $\psi(r) := \phi(u_r)$ (left), $Q(r) := \frac{\psi''(r)}{(\psi'(r))^2}$ (centre), and $H(r) = A\tanh(\frac{B}{A}\phi(u_r))$ with $A = 0.4$ and $B = 8$ (right).

numerical evaluations. One finds

$$\psi(r) = \frac{1}{r} + \frac{\log\left(\frac{4+4r-3r^2}{4-4r-3r^2}\right)}{\log\left(\frac{4-9r^2}{4-r^2}\right)} = \frac{r}{6} + \frac{7r^3}{40} + \frac{461r^5}{2016} + \frac{4619r^7}{13440} + \dots . \tag{A.2}$$

From this numerical evidence (see Figure 3 for a plot) it is clear that, for $r \in [0, 0.4]$,

$$\psi(r) \geqslant \frac{r}{6}, \quad \text{and}$$

$$\frac{\psi''(r)}{(\psi'(r))^2} = \frac{189\,r}{5} - \frac{12862\,r^3}{175} + \frac{44487\,r^5}{500} - \frac{346403009\,r^7}{4042500} + \dots \leqslant \frac{189}{5}r.$$

Hence $H'(r) = G'(\psi(r))\,\psi'(r) \leqslant G'(\frac{r}{6})\,\psi'(r)$. As

$$H'' = (G \circ \psi)'' = \left(\frac{G''}{G'} \circ \psi + \frac{\psi''}{(\psi')^2}\right) \cdot (\psi')^2 \cdot (G' \circ \psi)\ ,$$

$H''(r) \leqslant 0$ follows provided $\frac{G''(\psi(r))}{G'(\psi(r))} \leqslant -\frac{189}{5}r$. Therefore, assumption (2.12) is fulfilled, if

$$G'(x) \leqslant \frac{1}{\psi'(6x)} \quad \text{or if} \quad \frac{G''(x)}{G'(x)} \leqslant -\frac{189}{5} \cdot 6x \tag{A.3}$$

For $G(x) = A\tanh(\frac{B}{A}x)$, in which case $G'(x) = B/\cosh(\frac{B}{A}x)^2$ and $\frac{G''(x)}{G'(x)} = -2\frac{B}{A}\tanh(\frac{B}{A}x)$ this can be checked numerically. (Observe that $0 \leqslant A \leqslant 0.4$ and distinguish the cases $B = G'(0) \leqslant 6$ and $B > 6$.) For an illustration see the rightmost plot of $H(r)$ in Figure 3.

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